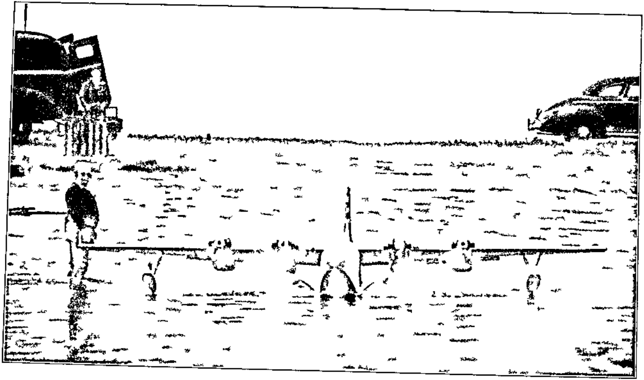
Dimensional Analysis
and
Theory of Models



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Dimensional Analysis and Theory of Models

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Preface

Dimensional analysis treats the general forms of equations that describe natural phenomena. Applications of dimensional analysis abound in nearly all fields of engineering, particularly in fluid mechanics and in heat-transfer theory.

A systematic and thorough treatment of the principles of dimensional analysis is undertaken in this book. The part of the presentation that is essential for practical applications presupposes little mathematical preparation, other than basic algebra and an understanding of the concept of a function. Buckingham's theorem, which really embodies all dimensional analysis, is introduced early, without the customary pi notation that is apt to render its meaning obscure to a beginner.

The application of dimensional analysis to any particular phenomenon is based on the assumption that certain variables, which are named, are the independent variables of the problem, and that all variables, other than these and the dependent variable, are redundant or irrelevant. This initial step—the naming of the variables—often requires a philosophic insight into natural phenomena.

The second step in the dimensional analysis of a problem is the formation of a complete set of dimensionless products of the variables. Most presentations of the subject demonstrate that the exponents of a dimensionless product are a solution of a certain set of homogeneous linear algebraic equations. Not just one solution of these equations is required, however, but rather, a fundamental system of solutions. This concept, and its application to dimensional analysis, are discussed in Chapter 3. Also, a routine numerical procedure for calculating a fundamental system of solutions of any set of homogeneous linear algebraic equations is presented. Often, a complete set of dimensionless products can be perceived without computation. However, even those who are experienced in dimensional analysis will occasionally find it expeditious to resort to a systematic method.

Chapter 4 presents a rigorous development of the theory of dimensional analysis. This chapter discloses the logic and the scope of the subject, and it supplies proofs of the theorems. This analysis is unavoidably too abstract and too mathematical to be readily assimilated by undergraduate engineering students. Therefore, some instructors will prefer to omit Chapter 4. This omission does not impede the understanding of the subsequent chapters.

The extensive use of small-scale models for investigating problems of engineering raises many important questions that are resolved by dimensional analysis A fairly complete account of the theories of similarity and model testing is presented in Chapter 5 A discussion of the many practical techniques in particular types of model tests is beyond the scope of this book

In the analysis of problems of stress and strain, dimensional analysis has perhaps received too little attention The results of computations or experiments can here be greatly expanded by means of dimensionless presentations Chapter 6 contains examples to illustrate this point

Questions of fluid motion have probably provided a greater incentive to the development of dimensional analysis than any other group of problems Some significant applications of dimensional analysis in this field are presented in Chapter 7

Another field in which dimensional analysis has been invaluable is the theory of heat transfer Applications of dimensional analysis to some thermal problems are illustrated in Chapter 8

In recent years dimensional analysis has had important applications in electromagnetic theory The ideas underlying the dimensions of electrical and magnetic entities are discussed in Chapter 9, and some applications are illustrated

Chapter 10 illustrates the method of deriving model laws from the differential equations that govern particular phenomena This method is frequently used in scientific literature

In order that students may understand the examples in the book and in order that they shall have a sufficiently mature outlook on science to understand properly the general subject matter it is advisable that they be familiar with principles of physics and engineering that are presented in the first three years of an engineering curriculum

The author s interest in dimensional analysis stems from his own investigations of the mathematical nature of the subject and from experience that he gained by teaching the subject for several years in a graduate course in fluid mechanics at the University of Illinois The book contains some novel features, particularly the method of computing dimensionless products that is presented in Chapter 3 and the algebraic theory that is presented in Chapter 4

Much credit for the work is owed to Professor Fred B Seely, who has taken a keen interest in the teaching of dimensional analysis, and who recommended to the author that his class notes be expanded into a book In the writing of the chapter on electrical applications Mr Knute J Takle, of the Naval Electronics Laboratory in San Diego California, rendered valuable assistance by contributing the illustrative examples, and by advis

ing the author on the systems of dimensions. The author is also indebted to Dr. Cevdet A. Erzen, Professor Will J. Worley, Mr. William B. Sanders, Jr., and Professor William Owen for suggestions and contributions that have added to the clarity and the interest of the work.

Unfortunately, the author did not learn, until after the page proofs were printed, that the inversion of the Stanton diagram (Figure 12) had been proposed previously. This type of chart was originally devised by S. P. Johnson, Preprinted Papers and Program, ASME Summer Meeting, June 1934, p. 98. The presentation of open-channel data in this form has been discussed by R. W. Powell, Resistance to Flow in Rough Channels, *Trans. Am. Geophys. Union*, Vol. 31, no. 4, Aug., 1950.

H. L. LANGHAAR

January 1951

Contents

CHAPTER 1

The Nature and the Use of Dimensions

The success of any physical investigation depends on the judicious selection of what is to be observed as of primary importance, combined with a voluntary abstraction of the mind from those features which, however attractive they may appear, we are not yet sufficiently advanced in science to investigate with profit.

J. CLERK MAXWELL

1. SCOPE OF DIMENSIONAL ANALYSIS

Dimensional analysis is a method by which we deduce information about a phenomenon from the single premise that the phenomenon can be described by a dimensionally correct equation among certain variables. The generality of the method is both its strength and its weakness. With little effort, a partial solution to nearly any problem is obtained. On the other hand, a complete solution is not obtained, nor is the inner mechanism of a phenomenon revealed, by dimensional reasoning alone.

The result of a dimensional analysis of a problem is a reduction of the number of variables in the problem. Students sometimes inquire, "What advantage is gained by this?" The answer to this question is apparent if one considers the labor that is required for the experimental determination of a function. A function of one variable may be plotted as a single curve. A function of two variables is represented by a family of curves (called a "chart"), one curve for each value of the second variable. A function of three variables is represented by a set of charts, one chart for each value of the third variable. A function of four variables is represented by a set of sets of charts, and so on. If, for example, five experimental points are required to plot a curve, twenty-five points are required to plot a chart of five curves, one hundred and twenty-five points are required to plot a set of five charts, etc. This situation quickly gets out of hand, particularly if each experimental point entails much expense, as is not unusual. Evidently, a reduction of the number of variables in a problem greatly amplifies the information that is obtained from a few experiments Consequently dimensional analysis has become an important mathematical tool of experimenters

2 Units of Force and Mass

If Newton s law $F = ma$ is applied to a freely falling body the force F is the weight W and the acceleration a is the acceleration of gravity g Consequently $W = mg$ or $m = W/g$ By setting $m = 1$ in this equation we perceive the following general rule

For consistency with the law $F = ma$ the weight of a unit mass must be exactly g units of force

The conventional systems of measurement conform to this rule Five common systems of measurement are discussed in the following

(a) *CGS System* The letters CGS denote respectively centimeter gram and second The gram is generally regarded as a unit of mass This is the thousandth part of a kilogram the latter mass being legally defined as the mass of a platinum cylinder that is deposited at the International Bureau of Weights and Measures at Sevres France The kilogram is very nearly the mass of a liter of water at 4°C By virtue of the law of the lever masses are accurately measured by a balance-type scale

For consistency with Newton s law the unit of force in the CGS system is defined to be that force which will give a gram mass an acceleration of 1 cm/sec^2 This unit of force is called a dyne Since the standard value of g is 980 665 cm/sec^2 the equation $W = mg$ shows that the weight of a gram mass on the earth is approximately 981 dynes

In the CGS system the unit of work is a dyne centimeter This unit is called an erg

(b) *MKS Mass System* The letters MKS denote respectively meter kilogram and second In the MKS mass system the kilogram is regarded as a unit of mass For consistency with the law $F = ma$ the unit of force is that force which will give a kilogram mass an acceleration of 1 m/sec^2 This unit of force is called a newton Evidently a newton is one hundred thousand dynes This is approximately 0 225 lb

The unit of work in the MKS mass system is a newton meter This unit is called a joule The joule is ten million ergs The watt is a unit of power that is defined to be one joule per second

The MKS mass system is used in electrical engineering It is regrettable that this simple system is not used by all engineers The widespread confusion about the units of force and mass would then be dispelled

(c) *MKS Force System* In the MKS force system the kilogram is regarded as a unit of force rather than a unit of mass The kilogram force is defined to be the weight of a kilogram mass under standard gravitational attraction. Consequently, a kilogram force is 980,665 dynes.

For consistency with the equation, $F = ma$, the unit of mass in the MKS force system is a "kilogram second squared per meter" (kg sec^2/m). This unit of mass has not received a special name. The equation, $W = mg$, shows that one kilogram second squared per meter is a mass that weighs approximately 9.81 kg force on the earth.

The MKS force system is extensively used in engineering practice in continental Europe.

(*d*) *British Mass System.* In the British mass system, the pound is regarded as a unit of mass. The pound mass is legally defined to be 0.4536 kg mass. For consistency with the equation, $F = ma$, the unit of force is defined to be that force which will give a pound mass an acceleration of one foot per second squared. This unit of force is called a "poundal." Since the standard acceleration of gravity is 32.174 ft/sec^2, the equation, $W = mg$, shows that the weight of a pound mass on the earth is approximately 32.2 poundals. Accordingly, one poundal is a force that is nearly equal to half an ounce.

The British mass system is frequently used in British technical writings.

(*e*) *American Engineering System.* Engineers in the United States usually regard the pound as a unit of force; namely, 0.4536 kg force. For consistency with the equation, $F = ma$, the unit of mass is then a "pound second squared per foot" (lb sec^2/ft). This unit of mass is called a "slug." The equation, $W = mg$, shows that the slug is a mass that weighs approximately 32.2 lb on the earth.

The American engineering system does not logically exclude the concept of a "pound of matter." A pound of matter may be defined to be the quantity of matter that weighs one pound on a spring scale. This is not an invariable quantity of matter, since it depends on the local acceleration of gravity. However, discrepancies between spring-type scales and balance-type scales, caused by variations of gravity, are ordinarily negligible.

For example, in hydraulics, "head" is interpreted to be energy: e.g. foot pounds of energy per pound of fluid. This interpretation is not reconcilable with the assertion that the unit of head is the foot, unless the pound is consistently regarded as a unit of force. Accordingly, in hydraulics, the pound is understood to be a unit of force, even though the expression "pound of fluid" is frequently used.

It must be admitted, however, that the use of a unit of force to express quantities of matter is misleading. Not only is the terminology deceptive, but the forms of equations also are affected. In commenting on some engineering formulas, Lord Rayleigh remarked, "When the question under consideration depends essentially upon gravity, the symbol g makes no appearance, but, when gravity does not enter into the question at all, g obtrudes itself conspicuously" An example to illustrate Rayleigh's point is the manner of writing Bernoulli's equation in terms of heads For a horizontal conduit, the equation is

$$\frac{p}{w} + \frac{V^2}{2g} = \text{constant}$$

Here, gravity has nothing to do with the flow, but the specific weight and the acceleration of gravity both appear in the equation The real nature of the phenomenon is displayed better if the equation is written,

$$p + \tfrac{1}{2}\rho V^2 = \text{constant} \quad \text{where} \quad w = \rho g$$

Note that ρ does not depend on gravity The latter form of the equation does not preclude the energy concept of hydraulics The terms in the equation may be interpreted to be energy per unit volume of fluid It is also permissible to divide the equation by ρ, in which case the terms represent energy per unit mass of fluid

Thermodynamicists and chemical engineers often interpret a pound of fluid to be a "pound mass," and they simultaneously designate the unit of pressure to be a "pound force per square foot" This dual interpretation of the pound is, to some extent, ambiguous, but it causes no discrepancies in practice, if Newton's law, $F = ma$, is not employed If Newton's law is employed, it must be expressed in the modified form, $F = kma$, in which the factor k compensates for the fact that the units are not compatible with the equation $F = ma$ When thermal and dynamical effects are both considered (as in the theory of compressible flow of gases), it is advantageous to employ distinct units of force and mass that conform to the equation, $F = ma$

3 Physical Dimensions

Scientific reasoning is based on concepts of various abstract entities, such as force, mass, length, time, acceleration, velocity, temperature, specific heat, and electric charge To each of these entities there is assigned a unit of measurement The entities mass, length, time, temperature, and electric charge, are, in a sense, independent of each other, for their units of measurement are prescribed by international standards Furthermore, the specified units of these entities determine the units of all other entities There is, however, nothing fundamental in the set of entities, mass, length, time, temperature, and electric charge A great many possibilities exist for choosing five mutually independent entities Frequently, the unit of force is considered to be prescribed, rather than the unit of mass When the units of force, length, and time are given, the unit of mass is uniquely determined by Newton's law, $F = ma$. In this case, the system of measurement is called a "force system," in contradistinction to the system in which the unit of mass is prescribed.

Even the fact that there are five mutually independent entities is a consequence of arbitrary definitions of the ways in which quantities are measured. For example, the kinetic theory of gases teaches that the temperature of a stationary gas is proportional to the mean kinetic energy of a single molecule of the gas. If the temperature of a stationary gas were defined to be exactly equal to the mean kinetic energy of a molecule, the unit of temperature would evidently be determined by the units of mass, length, and time.

J. Clerk Maxwell,[1] the Scottish physicist and philosopher, employed symbols of the type $[F]$, $[M]$, $[L]$, $[T]$, $[\Theta]$ to denote force, mass, length, time, and temperature, respectively. He formed products of powers of these symbols, which he called "dimensions." It is difficult to perceive what significance Maxwell attributed to dimensions, although he apparently believed that they would be useful for displaying analogies among various branches of physics, such as mechanics, electricity, and heat. Many controversies have been waged over the significance of dimensions. The polemics frequently display attitudes akin to Leibniz's feeling for the imaginary number, "A fine and wonderful recourse of the divine spirit, almost an amphibian between being and not being." But the general conclusion that emerges from the discussions is that the concept of dimensions is of little importance to philosophy. On the other hand, dimensions serve a mathematical purpose. *They are a code for telling us how the numerical value of a quantity changes when the basic units of measurement are subjected to prescribed changes.* This is the only characteristic of dimensions to which we need ascribe significance in the development of dimensional analysis. The use of dimensions for transforming units of measurement is illustrated in Article 4.

The dimensions of entities follow from definitions or from physical laws. For example, a velocity u is a derivative of distance with respect to time; i.e. $u = dx/dt$. Since dx is an increment of length and dt is an increment of time, the dimension of velocity is $[L/T]$ or $[LT^{-1}]$. Similarly, since acceleration is represented by a derivative du/dt, in which du is an increment of velocity, the dimension of acceleration is $[L/T^2]$ or $[LT^{-2}]$. These dimensions show that velocities may be expressed in feet per second, centimeters per second, miles per hour, etc., and that accelerations may be expressed

[1] J. Clerk Maxwell, On the Mathematical Classification of Physical Quantities, *Proc. London Math. Soc.*, Vol. III, no. 34, p. 224, Mar. 1871. See also J. Clerk Maxwell, *Theory of Heat*, Longmans, Green, London, 1894.

in feet per second squared, centimeters per second squared, miles per hour squared, etc

Since force and acceleration have the respective dimensions $[F]$ and $[LT^{-2}]$, Newton's law, $F = ma$, shows that mass has the dimension $[FT^2L^{-1}]$ Conversely, in the mass system, force has the dimension $[MLT^{-2}]$ To convert dimensions from the force system to the mass system, the dimensional expression $[MLT^{-2}]$ is accordingly substituted for $[F]$ The dimensional relationships among $[F]$ $[M]$, $[L]$, and $[T]$ indicate the following relationships among the corresponding units:

$$1 \text{ lb} = 1 \text{ slug ft/sec}^2, \qquad 1 \text{ dyne} = 1 \text{ g cm/sec}^2$$

$$1 \text{ slug} = 1 \text{ lb sec}^2/\text{ft}, \qquad 1 \text{ g} = 1 \text{ dyne sec}^2/\text{cm}$$

Here, the pound is considered to be a unit of force, and the gram is considered to be a unit of mass

The dimension of mass density is evidently $[ML^{-3}]$ or $[FL^{-4}T^2]$ Accordingly, in the American engineering system, mass density is expressed in the unit (slug/ft³) or (lb sec²/ft⁴)

Pressure and stress, being force per unit area have the dimension $[FL^{-2}]$ or $[ML^{-1}T^{-2}]$ The dynamic coefficient of viscosity μ is defined by the equation $\tau = \mu\, du/dy$ in which τ is a stress, du is an increment of velocity, and dy is an increment of length Consequently, the dimension of the dynamic coefficient of viscosity is $[FL^{-2}T]$ or $[ML^{-1}T^{-1}]$ The kinematic coefficient of viscosity ($\nu = \mu/\rho$) then has the dimension $[L^2T^{-1}]$

The unit of temperature may be assigned independently of the units of the entities of mechanics Consequently, the symbol $[\Theta]$ denoting temperature, is regarded as one of the fundamental dimensions

The dimension of an arbitrary variable z is denoted by $[z]$ If z is dimensionless, this fact may be denoted by $[z] = [1]$ The dimension of an integral $\int y\, dx$ is $[y\, dx]$ or $[y]\, [dx]$

There is no essential reason for having two systems of dimensions, the force system and the mass system As in many other instances, practices in various fields of pure and applied science have not been correlated well enough to insure the development of a universal convention Some writers have suggested that the difficulty may be circumvented by letting both $[F]$ and $[M]$ be fundamental dimensions This can be accomplished by writing Newton's law in the form $F = kma$ The factor k (which has the dimension $[FT^2/ML]$) would then appear in the formulas of dynamics It is a moot question whether the inconvenience of an intrusion of k's in all fields of dynamics would outweigh advantages that would result by dissociating the dimensions of force and mass

4. Transformation of Units of Measurement

Dimensional expressions enable us to convert units readily from one system of measurement to another. For example, to express an acceleration of 20 cm/sec² in the unit "miles per hour squared," set 1 cm = 6.2137×10^{-6} mi and 1 sec = 2.777×10^{-4} hr. Then,

$$20\left[\frac{\text{cm}}{\text{sec}^2}\right] = 20\left[\frac{6.2137 \times 10^{-6}\ \text{mi}}{(2.777 \times 10^{-4})^2\ \text{hr}^2}\right] = 1610\left[\frac{\text{mi}}{\text{hr}^2}\right]$$

The method illustrated in this example is perfectly general.

Frequently, empirical formulas contain coefficients that are valid only for a particular set of units. The above method may be used to modify the coefficients, so that different units may be used. The procedure is illustrated by the following example.

Example 1. Friction on the Wall of a Flume

The average shearing stress τ (lb/ft²) that a flowing liquid exerts on the wall of a concrete flume is assumed to be given by the empirical formula,

$$\tau = 0.0021\ \rho\ V^2 R^{-1/3}$$

in which ρ is the mass density of the liquid (slug/ft³), V is the average velocity of the liquid (ft/sec), and R is the ratio of the cross-sectional area to the wetted perimeter (hydraulic radius) in feet. It is desired to modify this formula so that it yields equivalent results when τ is expressed in kg/m², ρ is expressed in kg sec²/m⁴, V is expressed in m/sec, and R is expressed in meters.

The formula is of the form, $\tau = K\rho V^2 R^{-1/3}$. This equation shows that the dimension of K is $[L^{1/3}]$. The given value of K is accordingly 0.0021 ft$^{1/3}$. Since 1 ft = 0.3048 m, it follows

$$K = 0.0021\ [\text{ft}^{1/3}] = 0.0021\ [0.3048^{1/3}\ \text{m}^{1/3}] = 0.00141\ [\text{m}^{1/3}]$$

Consequently, when the formula is adapted to the new units, it becomes

$$\tau = 0.00141\ \rho V^2 R^{-1/3}$$

Note that the conversion of the force unit from the pound to the kilogram is immaterial, since K does not contain the dimension of force.

The result may be checked by assigning numerical values to ρ, V, and R; say $\rho = 2$ slug/ft³, $V = 10$ ft/sec, $R = 8$ ft. Then the original formula yields $\tau = 0.21$ lb/ft². In the metric system, the specified values are $\rho = 105.1$ kg sec²/m⁴, $V = 3.05$m/sec, $R = 2.44$ m. Hence, the new formula yields $\tau = 1.023$ kg/m². Since this is equivalent to 0.21 lb/ft², the two formulas agree.

A somewhat different application of the method of conversion of units is illustrated by the following example

Example 2 Astronomical System of Measurement

Newton's law of gravitation asserts that any two masses m and m' attract each other with a force F that is inversely proportional to the square of the distance r between the masses This law is expressed by the equation, $F = kmm'/r^2$ In the CGS system of measurement, the value of the gravitational constant is

$$k = 6.7 \times 10^{-8}\left[\frac{\text{cm}^3}{\text{g sec}^2}\right]$$

By definition, the astronomical system of measurement is such that $k = 1$ Letting the units of length and time be the mile and the second, let us express the astronomical units of force and mass in terms of familiar units

Denote the astronomical unit of mass by "asm" Set 1 g = X asm, and 1 cm = 6.2137×10^{-6} mi Then,

$$k = 6.7 \times 10^{-8}\left[\frac{\text{cm}^3}{\text{g sec}^2}\right] = 6.7 \times 10^{-8}\left[\frac{(6.2137 \times 10^{-6})^3\ \text{mi}^3}{(X\ \text{asm})\ \text{sec}^2}\right] = 1\left[\frac{\text{mi}^3}{\text{asm sec}^2}\right]$$

The last equation expresses the fact that $k = 1$ in the astronomical system It follows,

$$6.7 \times 6.2137^3 \times \frac{10^{-26}}{X} = 1$$

or

$$X = 1.607 \times 10^{-23}$$

Hence, $1\ \text{g} = 1.607 \times 10^{-23}$ asm or $1\ \text{asm} = 6.223 \times 10^{22}$ g

Therefore, $1\ \text{asm} = 4.264 \times 10^{18}$ slug

Thus, the astronomical unit of mass is expressed in terms of slugs The astronomical unit of force is now determined by Newton's equation, $F = kmm'/r^2$ For, let $m = m' = 1$ asm $= 6.223 \times 10^{22}$ g, $r = 1$ mi = 160,940 cm, and $k = 6.7 \times 10^{-8}$ cm^3/g sec^2 Then, since $k = 1$ in the astronomical system, the value of F is one astronomical unit of force (1 asf) In the CGS system, this force is

$$F = \frac{kmm'}{r^2} = \frac{6.7 \times 10^{-8}(6.223 \times 10^{22})^2}{(160{,}940)^2} = 1.003 \times 10^{28}\ \text{dyne}$$

or

$$F = 1\ \text{asf} = 2.25 \times 10^{22}\ \text{lb}$$

The results are summarized below:

1 astronomical unit of mass = 1 asm = 6.223×10^{22} g = 4.264×10^{18} slug

1 astronomical unit of force = 1 asf = 1.003×10^{28} dyne = 2.25×10^{22} lb

These relationships are valid only if the units of length and time are the mile and the second.

Example 3. Dimensions of Mass and Force in the Astronomical System

Max Planck,[2] the eminent physicist, has cited the astronomical system of measurement as an illustration of the conventionalism in dimensions. Since $k = 1$ in the astronomical system (see Example 2), Newton's law of gravitation is expressed by the equation, $F = mm'/r^2$. In conjunction with Newton's law of inertia, $F = ma$, this yields

$$[ma] = \left[\frac{mm'}{r^2}\right] \quad \text{or} \quad [MLT^{-2}] = [M^2L^{-2}]$$

Hence,

$$[M] = [L^3T^{-2}]$$

The equation, $F = ma$, now yields $[F] = [L^4T^{-4}]$. Accordingly, the units of mass and force in the astronomical system are, respectively, the "mile cubed per second squared" (mi^3/sec^2) and the "mile fourth per second fourth" (mi^4/sec^4). These relationships imply that the specified units of length and time determine the units of mass and force in the astronomical system. This is indeed true; in fact, the results of Example 2 may be expressed,

$$1 \text{ asm} = 1 \text{ mi}^3/\text{sec}^2 = 4.264 \times 10^{18} \text{ slug}$$

and

$$1 \text{ asf} = 1 \text{ mi}^4/\text{sec}^4 = 2.25 \times 10^{22} \text{ lb}$$

To those who attribute an intrinsic meaning to dimensions, the relationships $[M] = [L^3T^{-2}]$ and $[F] = [L^4T^{-4}]$ may seem paradoxical. However, these dimensions determine unequivocally how values of quantities in the astronomical system are changed, when the units of length and time are changed. If, for example, the kilometer is adopted as the unit of length, rather than the mile, the astronomical units of mass and force are changed. The conversion is calculated as follows:

$$1\left[\frac{\text{mi}^3}{\text{sec}^2}\right] = 1\left[\frac{1.6094^3 \text{ km}^3}{\text{sec}^2}\right] = 4.17\left[\frac{\text{km}^3}{\text{sec}^2}\right]$$

$$1\left[\frac{\text{mi}^4}{\text{sec}^4}\right] = 1\left[\frac{1.6094^4 \text{ km}^4}{\text{sec}^4}\right] = 6.71\left[\frac{\text{km}^4}{\text{sec}^4}\right]$$

[2] Max Planck, *General Mechanics* (Vol. I of *Introduction to Theoretical Physics*), Article 28, Macmillan, London, 1932.

Accordingly, when the units of length and time are the kilometer and the second, the astronomical units of mass and force are

$$1\ \text{km}^3/\text{sec}^2 = \frac{4\ 264}{4\ 17} \times 10^{18}\ \text{slug} = 1\ 022 \times 10^{18}\ \text{slug}$$

$$1\ \text{km}^4/\text{sec}^4 = \frac{2\ 25}{6\ 71} \times 10^{22}\ \text{lb} = 3\ 36 \times 10^{21}\ \text{lb}$$

In view of this example, and similar illustrations in other branches of physics Max Planck has remarked "To inquire into the 'real' dimension of a quantity has no more meaning than to inquire into the 'real' name of an object Many unfruitful controversies in physical literature, particularly those concerning the electromagnetic system of measurement, would have been avoided, had this circumstance always been properly appreciated"

It may be noted that Newton's law of gravitation is of the same form as Coulomb's law of force between electric charges The features that distinguish the astronomical system from the mass-length time system in dynamics consequently distinguish the Gaussian system from the Giorgi system in electromagnetic theory This matter is discussed in Article 60

TABLE 1

CONVERSION FACTORS

Units of Length

1 m = 39 37 in = 3 281 ft
1 ft = 30 480 cm = 0.30480 m
1 in = 2 54 cm
1 mi = 5280 ft = 1609 4 m = 160 940 cm
1 cm = 6 2137 × 10^{-6} mi

Units of Force and Mass

1 kg = 2 2046 lb	1 lb = 444 820 dyne = 4 4482 newton
1 lb = 0 4536 kg	1 slug = 14 594 g = 14 594 kg
1 metric ton = 1000 kg	1 kg (force) = 980 665 dyne
1 kg = 0 06852 slug	1 newton = 0 2248 lb = 100 000 dyne

Units of Temperature

1°C = 1 8°F

Units of Viscosity

1 poise = 1 g/cm sec = 1 dyne sec/cm^2 = 0 002088 slug/ft sec = 0 002088 lb sec/ft^2
1 stoke = 1 cm^2/sec = 0 001076 ft^2/sec

TABLE 2

DIMENSIONS OF ENTITIES

	Mass System	*Force System*
Length	$[L]$	$[L]$
Time	$[T]$	$[T]$
Temperature	$[\Theta]$	$[\Theta]$
Force	$[MLT^{-2}]$	$[F]$
Mass	$[M]$	$[FL^{-1}T^2]$
Specific Weight	$[ML^{-2}T^{-2}]$	$[FL^{-3}]$
Mass Density	$[ML^{-3}]$	$[FL^{-4}T^2]$
Angle	$[1]$	$[1]$
Pressure and Stress	$[ML^{-1}T^{-2}]$	$[FL^{-2}]$
Velocity	$[LT^{-1}]$	$[LT^{-1}]$
Acceleration	$[LT^{-2}]$	$[LT^{-2}]$
Angular Velocity	$[T^{-1}]$	$[T^{-1}]$
Angular Acceleration	$[T^{-2}]$	$[T^{-2}]$
Energy, Work	$[ML^2T^{-2}]$	$[FL]$
Momentum	$[MLT^{-1}]$	$[FT]$
Power	$[ML^2T^{-3}]$	$[FLT^{-1}]$
Moment of a Force	$[ML^2T^{-2}]$	$[FL]$
Dynamic Coefficient of Viscosity	$[ML^{-1}T^{-1}]$	$[FL^{-2}T]$
Kinematic Coefficient of Viscosity	$[L^2T^{-1}]$	$[L^2T^{-1}]$
Moment of Inertia of an Area	$[L^4]$	$[L^4]$
Moment of Inertia of a Mass	$[ML^2]$	$[FLT^2]$
Surface Tension	$[MT^{-2}]$	$[FL^{-1}]$
Modulus of Elasticity	$[ML^{-1}T^{-2}]$	$[FL^{-2}]$
Strain	$[1]$	$[1]$
Poisson's Ratio	$[1]$	$[1]$

PROBLEMS

1. Verify the conversion factors for the units of viscosity that are given in Table 1.

2. A mass is 10^{-5} ton hr^2/mi. What is the value of the mass expressed in slugs? In grams? What is the weight of the mass in pounds, under standard gravitational attraction? (1 ton = 2000 lb.)

3. Using the result of Example 2, calculate the mass of the earth in astronomical units. In slugs. In kg sec^2/m. Hence, compute the average specific gravity of the earth. (*Hint.* One slug weighs 32.17 lb at a distance of 3960 mi from the center of the earth.)

4. The dimension of a quantity is $[F^{1/2}L^{-3}T^{-2}\Theta^{-1/2}]$. By what factor is the numerical value of the quantity changed, if the unit of force is changed from megadynes to pounds (1 megadyne = 10^6 dyne), the unit of length is changed from meters to feet, the unit of time is changed from seconds to minutes, and the unit of temperature is changed from degrees centigrade to degrees Fahrenheit?

5. The dimension of a quantity is $[M^3L^{-1}T^2\Theta^{-1}]$. By what factor is the numerical value of the quantity changed if the units of measurement are subjected to the same changes as in Problem 4?

6. R. Iribarren Cav. has developed the following formula for the required weight of each armor rock in a rubble breakwater:

$$W = \frac{0.019\, sh^3}{(s-1)^3(\cos\alpha - \sin\alpha)^3}$$

in which W is the weight of a rock in metric tons, h is the height of the waves in meters, s is the specific gravity of the rock relative to sea water, and α is the angle of the face of the breakwater with respect to the horizontal. Modify this formula, so that it yields equivalent results when W is expressed in pounds and h is expressed in feet. Assuming that W is proportional to the specific weight w of the water, generalize the formula so that it is valid for any consistent units of measurement. Calculate W (lb) for the case $s = 2.5$, $h = 20$ ft, $w = 64$ lb/ft^3, and $\tan \alpha = 0.25$.

7. K. D. Wood has given (in 1935) the empirical formula,

$$C = (150 + 0.04V^2)A$$

in which C is the cost of a wind tunnel in dollars, V is speed of the air in mi/hr, and A is the area of the throat in square feet. Modify this formula so that it yields equivalent results when V is expressed in ft/sec and A is expressed in square inches. Check the result by a numerical example.

8. The mean rate of flow of air through a nozzle is approximated by Fliegner's formulas

$$M = 0.76A\sqrt{\frac{(p_1 - p_2)p_2}{\theta_1}} \quad \text{for} \quad p_2 > \tfrac{1}{2}p_1$$

$$M = 0.38A p_1 \theta_1^{-\frac{1}{2}} \quad \text{for} \quad p_2 < \tfrac{1}{2}p_1$$

in which M is the mass rate of flow in kg/sec, A is the area of the orifice in square centimeters, θ_1 is the absolute temperature in the vessel in degrees centigrade, and p_1 and p_2 are the internal and external pressures in kg/cm^2. Modify these formulas so that they yield equivalent results when M is expressed in slug/sec, A is expressed in square inches, θ_1 is expressed in degrees absolute Fahrenheit and p_1 and p_2 are expressed in lb/in.^{2} (*Hint*: Note that the kilogram is used simultaneously as a unit of force and a unit of mass.)

9. According to modern physics, matter is a form of energy. One gram mass is 9×10^{20} ergs of energy. Consider a system of measurement S, in which the unit of work is the energy of a gram mass. Let mass and time be fundamental dimensions, and let their units be the gram and the second. If Newton's equation $F = ma$ is retained, what are the dimensions of length, velocity, and force in system S? Prove that the unit of length in system S is the distance traveled by light in one second (Velocity of light = 3×10^{10} cm/sec). Express the unit of force of system S in dynes. In pounds.

CHAPTER 2

Principles and Illustrations of Dimensional Analysis

It happens not infrequently that results in the form of "laws" are put forward as novelties on the basis of elaborate experiments, which might have been predicted a priori after a few minutes consideration.

LORD RAYLEIGH

5. DIMENSIONAL HOMOGENEITY

An equation will be said to be *dimensionally homogeneous* if the form of the equation does not depend on the fundamental units of measurement. For example, the equation for the period of oscillation of a simple pendulum ($T = 2\pi\sqrt{L/g}$) is valid whether length is measured in feet, meters, or miles, and whether time is measured in minutes, days, or seconds. Therefore, by definition, the equation is dimensionally homogeneous. If the value $g = 32.2$ ft/sec^2 is substituted in the equation, there results $T = 1.11\sqrt{L}$. This equation is correct for pendulums on the earth, but it is no longer dimensionally homogeneous, since the factor 1.11 applies only if length is measured in feet and time is measured in seconds. It might be argued that the factor 1.11 itself has the dimension $[L^{-1/2}T]$. However, dimensions must not be assigned to numbers, for then any equation could be regarded as dimensionally homogeneous.

It can be deduced from the above definition of dimensional homogeneity that an equation of the form $x = a + b + c + \cdots$ is dimensionally homogeneous if, and only if, the variables $x, a, b, c, \cdots$ all have the same dimension. This theorem is proved in Chapter 4. It is a useful theorem for checking derivations. If a derived equation contains a sum or a difference of two terms that have different dimensions, a mistake has been made. This principle may be applied to differential equations and integral equations, as well as to algebraic equations. It should not be assumed, however, that an empirical equation is necessarily dimensionally homogeneous, 6 General Remarks on Dimensional Analysis

As compared to the general types of functions that are investigated in mathematical analysis, dimensionally homogeneous functions are a special class The theory of dimensional analysis is the mathematical theory of this class of functions This theory is purely algebraic It is developed in an abstract form in Chapter 4

The application of dimensional analysis to a practical problem is based on the hypothesis that the solution of the problem is expressible by means of a dimensionally homogeneous equation in terms of specified variables This hypothesis is justified by the fact that the fundamental equations of physics are dimensionally homogeneous and that relationships that are deducible from these equations are consequently dimensionally homogeneous However, we may not logically assume a priori that an unknown equation is dimensionally homogeneous, unless we know that the equation contains all the variables that would appear in an analytical derivation of the equation For example, in the problem of drag on a spherical body in an air stream, it might be argued that the density and the viscosity may be disregarded, since they are constants for standard air The equation for the drag force F would then be of the form $F = f(V, D)$, in which V is the velocity of the stream and D is the diameter of the body However, it is obviously impossible to construct a dimensionally homogeneous equation of this form, since the variables V and D do not contain the dimensions of force or mass

The first step in the dimensional analysis of a problem is to decide what variables enter the problem If variables are introduced that really do not affect the phenomenon too many terms may appear in the final equation If variables are omitted that logically may influence the phenomenon, the calculations may reach an impasse but, more often, they lead to an incomplete or erroneous result Even though some variables are practically constants (e g the acceleration of gravity) they may be essential because they combine with other active variables to form dimensionless products

Frequently the question arises, How do we know that a certain variable affects a phenomenon?" To answer this question, one must understand enough about the problem to explain *why* and *how* the variable influences the phenomenon Before one undertakes the dimensional analysis of a problem, he should try to form a theory of the mechanism of the phenomenon Even a crude theory usually discloses the actions of the more important variables If the differential equations that govern the phenomenon are available, they show directly which variables are significant

There are some fields in which dimensional analysis has had little application, because the existing knowledge in these fields is inadequate to indicate the significant variables. For example, the endurance limits of members that are subjected to alternating stresses have not been correlated with other measurable properties of materials. Consequently, dimensional analysis cannot yet be brought to bear on questions of fatigue of materials.

The nature of dimensional analysis will be made clearer by an example. Consider a smooth spherical body of diameter D that is immersed in a stream of incompressible fluid. Let the velocity of the stream at some distance ahead of the body be V. Then the drag force F on the body is represented by an equation of the form, $F = f(V, D, \rho, \mu)$, in which ρ is the mass density of the fluid, μ is the dynamic coefficient of viscosity of the fluid, and f represents an unspecified function. This equation merely means that F *depends on* the variables V, D, ρ, and μ. Nothing is said about the nature of the dependency. It is shown in Article 8 that, in order for the equation to be dimensionally homogeneous, it must have the following form:

$$\frac{F}{\rho V^2 D^2} = f_1\left(\frac{VD\rho}{\mu}\right)$$

The function f_1 is unknown, but it depends on only one variable $VD\rho/\mu$, rather than on the four separate variables V, D, ρ, and μ. Observe that the expressions $F/\rho V^2 D^2$ and $VD\rho/\mu$ are dimensionless. Expressions of this type are called *dimensionless products*. In general, if L denotes a length, the dimensionless product $VL\rho/\mu$ or VL/ν is called *Reynolds' number*. Reynolds' number is conventionally denoted by $\mathbf{R}$ or N_R. The dimensionless product $F/\rho V^2 L^2$ is called a *pressure coefficient*, since F/L^2 may be interpreted to be a pressure.

The projected area of a sphere is $\frac{1}{4}\pi D^2$. Consequently, the preceding equation for the drag on a sphere may be written

$$\frac{F}{\rho V^2 A} = \frac{1}{2}\,\frac{8}{\pi} f_1(\mathbf{R})$$

The term $(8/\pi)f_1(\mathbf{R})$ is called the *drag coefficient*. It is denoted by C_D. Accordingly, the equation for the drag on a sphere may be written,

$$F = \tfrac{1}{2} C_D \rho V^2 A \tag{1}$$

Since C_D is a function of $\mathbf{R}$, we may plot a graph in which the abscissa is $\mathbf{R}$ and the ordinate is C_D. Figure 1 is an experimental graph of this relationship for smooth spherical bodies. The curve is plotted to a logarithmic scale, since otherwise the falling part of the curve at the left side of the graph would be crowded very close to the vertical axis.

Figure 1 gives complete information concerning the drag forces on smooth spherical bodies of all sizes in an incompressible fluid with any density and any viscosity and with any speed of flow. To provide the same information without a dimensional analysis of the problem would require about twenty five charts that would show separately the effects of each of the variables V, D, ρ, and μ. Figure 1 remains approximately valid for a compressible fluid, such as air, if the speed of flow is less than half the speed of sound in the fluid. It is found that the location of the sudden drop in the curve, which is due to the development of turbulence in the boundary layer, depends on the initial turbulence in the fluid stream.

Any point on Figure 1 could be obtained by a test of a model. Suppose, for example, that the prototype is a smooth sphere 10 ft in diameter that is immersed in air at 60°F with velocity 50 ft/sec. The cost of a testing

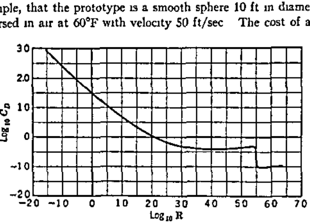

FIG 1 Drag Coefficient for Smooth Spheres

Reference Das Widerstandsproblem, F Eisner, *Proc 3d Intern Congr Applied Mechanics* Stockholm, 1931

apparatus to measure directly the drag under these conditions would probably be prohibitive. However the value of C_D can be obtained by testing a model sphere 1 ft in diameter in *water* at 60°F with speed of flow equal to 38 ft/sec, since Reynolds number for the model sphere, under the specified conditions, is the same as for the prototype.

The preceding example illustrates two important properties of dimensionless graphs.

1 A dimensionless graph provides much more information than a graph in which the coordinates have dimensions.

2 Points on dimensionless graphs can frequently be determined by tests of models.

7 COMPLETE SETS OF DIMENSIONLESS PRODUCTS

The commonest variables in fluid mechanics are force F, length L, velocity V, mass density ρ, dynamic coefficient of viscosity μ, acceleration of gravity g, speed of sound c, and surface tension σ. The following dimensionless products can be formed from these variables:

Reynolds' Number $\quad \mathbf{R} = \dfrac{VL\rho}{\mu} = \dfrac{VL}{\nu}. \quad \left(\nu = \dfrac{\mu}{\rho}\right)$

Pressure Coefficient $\quad \mathbf{P} = \dfrac{F}{\rho V^2 L^2} = \dfrac{p}{\rho V^2}. \quad (p = \text{pressure})$

Froude's Number $\quad \mathbf{F} = \dfrac{V^2}{Lg}$

Mach's Number $\quad \mathbf{M} = \dfrac{V}{c}$

Weber's Number $\quad \mathbf{W} = \dfrac{\rho V^2 L}{\sigma}$

In a gas, the pressure p, the density ρ, and the speed of sound c are related by the equation, $c = \sqrt{\gamma p/\rho}$, in which γ is a dimensionless constant ($\gamma = 1.40$ for air and other diatomic gases). Therefore, only two of the variables p, ρ, and c need be included in problems of gas dynamics. If c is omitted, Mach's number may be written in the form, $\mathbf{M} = V\sqrt{\rho/\gamma p}$. Aside from the factor γ, this is a power of the pressure coefficient. Accordingly, in gas dynamics, it is always possible to substitute a pressure coefficient for Mach's number.

Innumerable dimensionless products can be formed from the variables $F, L, V, \rho, \mu, g, c, \sigma$. However, it follows from a theorem that is derived in Chapter 4 that any dimensionless product of these variables is of the form,

$$\mathbf{R}^{a_1}\mathbf{P}^{a_2}\mathbf{F}^{a_3}\mathbf{M}^{a_4}\mathbf{W}^{a_5}$$

in which a_1, a_2, a_3, a_4, a_5 are constant exponents. On the other hand, the products **R**, **P**, **F**, **M**, **W** are independent of each other, in the sense that no one of these products is a product of powers of the others. This is obvious from the fact that μ occurs only in **R**, F occurs only in **P**, g occurs only in **F**, c occurs only in **M**, and σ occurs only in **W**.

Examples of other dimensionless products that can be formed from the given variables are $V^3\rho/\mu g$ and $\rho F/\mu^2$. However, these are not new products, since they are expressible in terms of the preceding products as follows:

$$\frac{V^3\rho}{\mu g} = \mathbf{RF}, \qquad \frac{\rho F}{\mu^2} = \mathbf{R}^2\mathbf{P}$$

The preceding discussion suggests the following general definition:

A set of dimensionless products of given variables is complete, if each product in the set is independent of the others, and every other dimension-

less product of the variables is a product of powers of dimensionless products in the set

Accordingly, (**R**, **P**, **F** **M**, **W**) is a complete set of dimensionless products of the variables (F, L, V, ρ, μ, g, c, σ)

In most problems of fluid mechanics, certain of the variables (F, L, V, ρ, μ, g, c, σ) are absent The dimensionless products (**R**, **P**, **F**, **M**, **W**), however, usually determine a complete set of dimensionless products of a subset of the variables If, for example, surface tension is insignificant, Weber's number is discarded, if g has no influence, Froude's number is discarded and so forth

A physical problem may involve several variables of the same kind Then the ratio of any two of these variables is a dimensionless product Mach s number is a dimensionless product of this type As another example suppose that two different fluids (e g air and water) enter into a phenomenon Then the ratio of densities and the ratio of viscosities of the two fluids are dimensionless products

A routine procedure for calculating a complete set of dimensionless products of any given set of variables is presented in Chapter 3 However, in some instances calculation can be avoided by using the standard products that are listed above

8 Buckingham s Theorem

Evidently any equation that relates dimensionless products is dimensionally homogeneous, i e the form of the equation does not depend on the fundamental units of measurement This observation may be formally stated as follows

> A sufficient condition that an equation be dimensionally homogeneous is that it be reducible to an equation among dimensionless products

E Buckingham[3] inferred the fundamental principle that the conditions of this theorem are also necessary Buckingham s theorem is accordingly stated as follows

> *If an equation is dimensionally homogeneous it can be reduced to a relationship among a complete set of dimensionless products*

This theorem is, by no means, self-evident An algebraic proof of the theorem is given in Chapter 4 Buckingham himself did not rigorously prove the theorem, although he presented evidence to make its truth seem plausible

[3] E Buckingham, On Physically Similar Systems Illustrations of the Use of Dimensional Equations *Phys Rev* Vol IV no 4 p 345 1914

Buckingham's theorem summarizes the entire theory of dimensional analysis. However, principles of dimensional analysis were employed before this theorem was expounded. As early as 1899, Lord Rayleigh[4,5] made an ingenious application of dimensional analysis to the problem of the effect of temperature on the viscosity of a gas (Example 8). Rayleigh's method is outwardly different from Buckingham's method, but it accomplishes the same results. To illustrate Rayleigh's method, let us again consider the drag force F that a smooth spherical body experiences in a stream of incompressible fluid. Consider tentatively a relationship of the form,

$$F = V^a D^b \rho^c \mu^d \tag{a}$$

The exponents must be adjusted to make the equation dimensionally homogeneous. This leads to the more special form,

$$F = \rho V^2 D^2 \mathbf{R}^n \tag{b}$$

where $\mathbf{R} = VD\rho/\mu$, and n is a numerical exponent.

Equation a is of such a restricted form that it cannot be expected to represent the phenomenon. However, Rayleigh pointed out that special solutions of the type of Equation b may be summed to give more general solutions. Accordingly, a general type of dimensionally homogeneous relationship is

$$F = \rho V^2 D^2 \sum_1^\infty A_n \mathbf{R}^n$$

in which the coefficients A_n are dimensionless constants. Since the series is a general function of $\mathbf{R}$, the solution is of the form,

$$F = \rho V^2 D^2 f(\mathbf{R})$$

in which f is an unspecified function.

Let us now consider the problem in the light of Buckingham's theorem. We assume only that the five variables are related by a dimensionally homogeneous equation. This may be indicated by $f(F, V, D, \rho, \mu) = 0$, in which f is an unspecified function. Buckingham's theorem asserts that, since the equation is dimensionally homogeneous, f is not actually a function of the five separate variables, but rather a function of a complete set of dimensionless products of the variables. According to the results of the preceding article, a complete set of dimensionless products of the variables is comprised of the pressure coefficient, $\mathbf{P} = F/\rho V^2 D^2$, and the Reynolds number, $\mathbf{R} = VD\rho/\mu$. Hence, by Buckingham's theorem, the equation

[4] Lord Rayleigh, On the Viscosity of Argon as Affected by Temperature, *Proc. Roy. Soc. London*, Vol. LXVI, pp. 68–74, 1899–1900.

[5] Lord Rayleigh, The Principle of Similitude, *Nature*, Vol. 95, 1915.

is reducible to the form, $f(P, R) = 0$. This relationship may be indicated in the explicit form, $P = f(R)$. If f is regarded merely as a symbol for some function, the relationships, $f(P, R) = 0$ and $P = f(R)$, mean the same thing, namely, that it is possible to plot a curve that shows the relationship between P and R. This is essentially the curve that is plotted in Figure 1. The equation, $P = f(R)$, is the same result that was obtained above, by Rayleigh's method. The reasoning that has led to the conclusion, $P = f(R)$, is not restricted to spherical bodies; it is valid for a body of any shape—for example an airplane wing. The form of the curve that relates P to R depends, of course, on the shape of the body. Dimensional analysis provides no information concerning the form of the curve.

Rayleigh's method of dimensional analysis does not differ intrinsically from Buckingham's method. The algebraic steps in the two methods are essentially the same. However, Buckingham's method absolves us from the indiscriminate use of infinite series. Too often, it is not explained that the construction of an infinite series is a logically indispensable step in Rayleigh's method. Consequently, the impression is created that the dependent variable in a physical problem may be arbitrarily equated to a product of powers of the independent variables and a numerical coefficient. This assumption is sometimes a legitimate approximation, particularly in problems of heat transfer, but it is not an essential part of dimensional analysis.

If n variables are connected by an unknown dimensionally homogeneous equation, Buckingham's theorem allows us to conclude that the equation can be expressed in the form of a relationship among $n - r$ dimensionless products, in which $n - r$ is the number of products in a complete set of dimensionless products of the variables. In Article 10 it is explained how to compute the number r. In most cases, r is equal to the number of fundamental dimensions in the problem. However this cannot be an infallible rule, since the number of fundamental dimensions in a problem may depend on the system of fundamental dimensions that is used. For example, problems of stress analysis usually involve only two dimensions, $[F]$ and $[L]$. However since $[F] = [MLT^{-2}]$ there are three dimensions if the mass system is used.

Example 4. Drag on a Ship

The drag force that the water exerts on a ship naturally depends on the shape of the hull. However, dimensional analysis is of little use for predicting the ways in which phenomena are affected by intricate shapes. Consequently, in dimensional analysis, it is often convenient to eliminate the consideration of shape effects by restricting attention to bodies of the same shape, i.e. bodies that are *geometrically similar*. If the shape of the hull of a ship is considered to be fixed from the outset, the hull is completely specified by its size. This may be designated by a single length L—say the length of the hull. If the hulls that are considered are required to have corresponding water lines, the length L also determines the draft (submerged depth). In this case, the drag force F depends on the length L, the speed V of the ship, the viscosity μ of the water, the mass density ρ of the water, and the acceleration of gravity g. The term g is important, since a large part of the energy that is used to propel the ship is dissipated in waves, and the energy of the waves depends on g.

In view of the preceding remarks, there is an equation of the form,

$$f(F, V, L, \rho, \mu, g) = 0$$

A complete set of dimensionless products consists of the pressure coefficient, $\mathbf{P} = F/\rho V^2 L^2$, the Reynolds number, $\mathbf{R} = VL\rho/\mu$, and the Froude number, $\mathbf{F} = V^2/Lg$. Consequently, by virtue of Buckingham's theorem, the above equation reduces to $f(\mathbf{P}, \mathbf{F}, \mathbf{R}) = 0$ or $\mathbf{P} = f(\mathbf{F}, \mathbf{R})$. This equation may be written

$$F = \rho V^2 L^2 f(\mathbf{F}, \mathbf{R})$$

The maximum cross-sectional area A of the portion of the hull that is below the water line is proportional to L^2—the constant of proportionality being determined by the shape of the hull. Consequently, there is a dimensionless constant k, such that $L^2 = \frac{1}{2}kA$. Substituting this in the above equation and setting $kf(\mathbf{F}, \mathbf{R}) = C_D$, we get

$$F = \tfrac{1}{2}C_D \rho V^2 A \qquad (2)$$

This is of the same form as the formula for the drag on a totally immersed body (Equation 1), but, in the present case, the drag coefficient C_D depends not only on Reynolds' number, but also on Froude's number.

The drag coefficient of a ship may be written in the form,

$$C_D = C'_D + C''_D$$

in which C'_D is the drag coefficient due to the shearing resistance of the water on the hull, and C''_D is the drag coefficient due to inequalities of pressure on the bow and the stern. The drag forces corresponding to these coefficients are, respectively, called "skin friction" and "form drag."* The skin friction is strongly influenced by viscosity, but it is practically independent of the wave pattern. Therefore, C'_D is a function of Reynolds' number alone. On the other hand, C''_D is practically independent of Reynolds' number. However, because of the energy of the waves, C''_D depends essentially on Froude's number.

* Wave resistance is commonly differentiated from form drag.

By means of boundary layer theory the skin friction drag of a ship may be computed Although the theory for computing the form drag coefficient is inadequate this coefficient is determined by a model test if the Froude number of the model equals the Froude number of the prototype Thus by a combination of theory and experiment the drag of a proposed hull can be reliably predicted

EXAMPLE 5 PRESSURE DROP IN A UNIFORM PIPE

The pressure drop Δp of liquid in a horizontal uniform pipe depends on the length L in which the pressure drop occurs the diameter D of the pipe

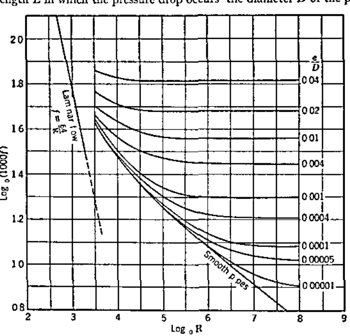

FIG 2 Pipe Friction Factors

Reference L F Moody *Trans ASME* Vol 66 p 671 Nov 1944 By courtesy of the American Society of Mechanical Engineers

the average velocity V of the fluid the viscosity μ the mass density ρ and the average height e of the surface roughness The interpretation of the roughness height e is a statistical problem that has not been satisfactorily solved However this problem need not be considered here The relationship among the variables is indicated by the equation

$$f(\Delta p \quad L \quad D \quad e \quad V \quad \rho \quad \mu) = 0$$

By Buckingham's theorem, this equation is expressible in terms of a complete set of dimensionless products. A complete set of dimensionless products of the variables consists of the pressure coefficient, $\mathbf{P} = \Delta p/\rho V^2$, the Reynolds number, $\mathbf{R} = VD\rho/\mu$, and the ratios of lengths, L/D and e/D. Consequently, there is a function F, such that

$$F\left(\mathbf{P}, \mathbf{R}, \frac{L}{D}, \frac{e}{D}\right) = 0$$

This may be written in the explicit form,

$$\mathbf{P} = F\left(\mathbf{R}, \frac{L}{D}, \frac{e}{D}\right) \tag{a}$$

The pressure drop in a uniform horizontal pipe is determined by the shearing stress at the wall, and this, in turn, is determined by the velocity profile. Therefore, the shearing stress is constant. It follows that the pressure drop Δp is proportional to the length L. Consequently, Equation a must take the more special form,

$$\Delta p = \tfrac{1}{2}\rho V^2 \frac{L}{D} f\left(\mathbf{R}, \frac{e}{D}\right) \tag{3}$$

Equation 3 is a form of the Darcy formula. The function $f(\mathbf{R}, e/D)$ is known as the *pipe friction factor.*

For a fixed value of the parameter e/D, it is possible to plot a curve which represents f as a function of $\mathbf{R}$. A chart that presents curves of this type for various values of e/D is known as a "Stanton diagram." Any curve of a Stanton diagram can be plotted from data obtained by measuring the flow of water in a single pipe, since $\mathbf{R}$ can be varied by varying the velocity of the water. Figure 2 is a Stanton diagram that was constructed by L. F. Moody. He recommends the following roughness heights for commercial pipes:

Type of Pipe	*Roughness Height e*
Drawn Tubing	5×10^{-6} ft.
Steel and Wrought Iron	0.00015 ft
Asphalted Cast Iron	0.0004 ft
Galvanized Iron	0.0005 ft
Plain Cast Iron	0.00085 ft
Concrete	0.001 ft to 0.01 ft
Riveted Steel	0.003 ft to 0.03 ft

Tests have indicated that a large riveted conduit possesses no single characteristic roughness height. Undoubtedly, two different roughness heights enter this problem—one representing the natural surface roughness and the other representing the rivet heads

EXAMPLE 6 CRITICAL REYNOLDS' NUMBER FOR FLOW IN PIPES

It is readily shown by dimensional analysis that Reynolds' number is the criterion that determines whether or not the flow in a pipe is turbulent For, if there is a critical velocity V_{cr} at which the transition from laminar flow to turbulence occurs, this velocity depends on the pipe diameter D, the viscosity μ, and the mass density ρ Consequently, there is a relationship of the form,

$$f(V_{cr}, D, \rho, \mu) = 0$$

The only dimensionless product that can be formed from the four variables is Reynolds' number, $R_{cr} = V_{cr}D\rho/\mu$, or some power of this number Therefore, by Buckingham's theorem, the equation must reduce to

$$f(R_{cr}) = 0$$

The solution of this equation is of the form, R_{cr} = constant It has been determined experimentally that the constant is about 2000, i e , the flow is laminar if $R < 2000$ and turbulent if $R > 2000$

Reynolds discussed this problem in essentially the same way as it is presented here.

EXAMPLE 7 VIBRATION OF A STAR*

A star, being a liquid body that is held together by its own gravity, may vibrate in various ways Of especial theoretical importance are vibrations in which all particles execute simple harmonic motions that are in phase with each other In this case, the particles simultaneously pass through their neutral positions and simultaneously reach their extreme positions A vibration of this type is called a *natural mode* The simplest natural mode of a vibrating star is a motion in which the surface alternately assumes oblate and prolate forms that are symmetrical with respect to a fixed axis Any small vibration of a frictionless system is a superposition of natural modes

Corresponding to any natural mode, there is a definite frequency n (i e number of vibrations per unit time) It is known that a small amount of viscosity does not significantly affect the frequency of a system Consequently, viscosity will be neglected Furthermore, for simplicity, it will

* The treatment of this problem by dimensional analysis was proposed by Rayleigh (Reference 5)

be assumed that the mass density ρ of the star is constant.* Then the only variables that can affect the frequency n of a given mode of vibration are the diameter D of the star, the density ρ, and the gravitational constant k. The constant k is significant, since it occurs in the mathematical derivation of the frequency of a system with a gravitational restoring force, if the units of mass, length, and time are specified independently. The dimension of k is $[M^{-1}L^3T^{-2}]$ (see Example 2).

The relationship among the variables is expressed by an equation of the form,

$$f(n, D, \rho, k) = 0 \tag{a}$$

The only dimensionless product of the variables is $n^2/k\rho$, or some power of this product. Consequently, D cannot enter the problem.

In view of Buckingham's theorem, Equation a now reduces to

$$f\left(\frac{n^2}{k\rho}\right) = 0, \quad \text{whence} \quad \frac{n^2}{k\rho} = C^2, \quad \text{a constant}$$

It follows that

$$n = C\sqrt{k\rho} \tag{b}$$

Thus, it is shown that the frequency of any natural mode of vibration of a star is independent of the diameter and directly proportional to the square root of the mass density.

The same conclusions can be deduced by using the astronomical system of measurement (see Example 2). Since, by definition, the gravitational constant is unity in the astronomical system, Equation a takes the simpler form,

$$f(n, D, \rho) = 0$$

In the astronomical system, the dimension of ρ is $[T^{-2}]$. Consequently, n^2/ρ is a dimensionless product. Since it is impossible to form a dimensionless product that contains D, the diameter again drops out of the problem. It follows,

$$f\left(\frac{n^2}{\rho}\right) = 0, \quad \text{whence} \quad \frac{n^2}{\rho} = (C')^2, \quad \text{a constant}$$

Hence,

$$n = C'\sqrt{\rho}$$

This equation is identical to Equation b, if we set $C' = C\sqrt{k}$.

In general, any product that is dimensionless in the mass-length-time system (e.g., Reynolds' number, Froude's number, etc.) is also dimension-

* The conclusions are the same if stars of variable density are considered, provided that attention is restricted to stars with similar mass distributions. In the case of variable density, ρ may be interpreted to be the density at the center of a star.

less in the astronomical system However, the converse is not true For example, $V^2/\rho L^2$ is a dimensionless product in the astronomical system (in which V = velocity, ρ = mass density, and L = length) Consequently, the astronomical system admits more possibilities for dimensionless products than the mass length time system Therefore, if the astronomical system is used in the dimensional analysis of a problem, the result is sometimes less specific than it would be if the mass length-time system were used This circumstance merely reflects the fact that the condition of dimensional homogeneity is more restrictive in the mass length time system than in the astronomical system—a natural consequence of the fact that there are only two dimensions in the astronomical system

Problems

1 If laminar flow exists between fixed horizontal concentric cylindrical walls with radii a and b the velocity u at the radius r is given by one of the following formulas

$$u = \frac{K}{4\mu}\left[b^2 - r^2 - (b^2 - a^2)\frac{\log\frac{b}{r}}{\log\frac{b}{a}}\right]$$

$$u = \frac{4\mu}{K}\left[b^2 - r^2 - (b^2 - a^2)\frac{\log\frac{b}{r}}{\log\frac{b}{a}}\right]$$

in which μ is the dynamic coefficient of viscosity and K is the pressure drop per unit length Determine by the dimensions of the quantities which formula is correct

2 If uniform flow in a horizontal pipe is laminar the pressure drop Δp in a length L does not depend on the mass density of the fluid since the particles of fluid are not accelerated Using this fact and the condition that Δp is proportional to L derive the most general form of a dimensionally homogeneous equation for Δp

3 Prove by dimensional analysis that the period of a frictionless pendulum of any shape is inversely proportional to the square root of the acceleration of gravity and that the period does not depend on the mass of the pendulum

4 Prove by dimensional analysis that the centrifugal force of a particle is proportional to its mass, proportional to the square of its velocity and inversely proportional to the radius of curvature of its path

5 If the depth of liquid in a flume is less than a certain value y_{cr} a hydraulic jump is possible For a flume with a V shaped cross section, y_{cr} depends on the rate of flow Q (volume per unit time), the acceleration of gravity g and the angle α between the walls of the flume Derive the general form of the equation for y_{cr} by dimensional analysis

6 Solve Problem 5 for a flume of rectangular cross section, assuming that y_{cr} is determined by the acceleration of gravity and the ratio, $q = Q/b$, in which b is the width of the flume

7 The ultimate bending moment M of a beam of rectangular cross section depends on the width b of the cross section, the depth h of the cross section, and the yield stress

σ_y of the material. Derive the most general form of a dimensionally homogeneous equation for M. Derive the more special form that the equation must take, by virtue of the fact that M is proportional to b. Then, how does M vary with h?

✓8. The pressure p at a stagnation point in an air stream depends on the pressure p_0 in the free stream, the mass density ρ_0 of the air in the free stream, and the velocity V of the stream. On what dimensionless product does the ratio p/p_0 depend?

✓9. On what dimensionless variables does the drag coefficient of an airplane wing depend, if the drag is affected by the size of the wing, the angle of attack, the speed of flight, the viscosity and density of air, and the speed of compression waves in air?

10. The pressure drop Δp due to a valve, elbow, orifice, or other obstruction in a pipe depends on the shape of the obstruction, the diameter D of the pipe, the average velocity V of the liquid in the pipe, the mass density ρ of the liquid, and the dynamic viscosity μ of the liquid. Determine the most general form of a dimensionally homogeneous equation for Δp. Determine the more special form that the equation takes if the effect of viscosity is negligible. Then how does Δp vary with D?

11. The height h that a liquid will rise in a capillary tube is inversely proportional to the diameter D of the tube. How does h vary with the surface tension σ and with the specific weight w?

12. The height h of the tide that is caused by a steady wind blowing over a lake depends on the average depth D of the lake, the length L of the lake, specific weight w of water, and the shearing stress τ of the wind on the water. What is the most general form of a dimensionally homogeneous equation for h?

13. The velocity V of a gas issuing from an orifice in a tank depends on the pressure p_0 and the mass density ρ_0 of the gas in the tank, and the pressure p_1 outside of the tank. For certain values of p_0 and p_1, the velocity of air issuing from an orifice is 300 ft/sec. For the same values of p_0 and p_1, what would the velocity be, if the tank contained hydrogen, rather than air? (The ratio of the density of air to the density of hydrogen is 14.4.)

14. The speed of sound in a gas depends on the pressure and the mass density. Prove by dimensional analysis that the speed of sound is proportional to the square root of the pressure and inversely proportional to the square root of the mass density.

15. The speed of sound in an elastic solid depends on the modulus of elasticity and on the mass density. How does it vary with the modulus of elasticity? With the mass density?

16. The frequency n of any natural mode of vibration of an elastic structure, in a class of geometrically similar structures, depends on a length L that specifies the size of the structure, the modulus of elasticity E, and the mass density ρ of the material. How does n vary with L? With E? With ρ?

17. Neglecting viscosity, surface tension, and wave amplitude, prove that the speed of progressive waves in deep water is proportional to the square root of the wave length. (Rayleigh)

18. The efficiency η of a power transmission consisting of two meshed gears depends on the diameters D and d of the gears, the dynamic viscosity μ of the lubricant, the angular speed N of the driving shaft, and the tooth load F per unit width. Make a dimensional analysis of the problem.

19. The weight W of a drop of liquid that falls slowly from a tube depends on the diameter D of the tube, the specific weight w of the liquid, and the surface tension σ of the liquid. What is the most general form of a dimensionally homogeneous equation for W? (Rayleigh)

20. Neglecting the effect of amplitude, prove that the period of any natural mode of oscillation of a frictionless liquid in a deep vertical cylindrical can that is open at the top is directly proportional to the square root of the diameter of the can. (Rayleigh)

21 The frequency n at which eddies are shed from a partially opened gate valve depends on the diameter D of the conduit, the average velocity V of the fluid in the conduit, the dynamic viscosity μ of the fluid and the mass density ρ of the fluid Derive the most general form of a dimensionally homogeneous equation for n

22 The frequency of a string that is forced to vibrate by the wind is 512 vibrations per second, if the wind has a certain velocity V What is the frequency, if the diameter of the string is doubled and the speed of the wind is halved? (The diameter is the only characteristic of the string that affects the frequency)

23 Assume that the velocity V of a progressive wave in a uniform stretched string depends on the tension T in the string and the mass m per unit length and that it is independent of the amplitude of the wave Can it then be deduced by dimensional analysis that V does not depend on the wave length? How does V vary with T? With m?

24 The volume Q of water that flows over a spillway per second, per foot of length along the crest of the spillway, depends on the height h of the water surface above the crest of the spillway, the acceleration of gravity g, a length L that specifies the size of the cross section of the spillway, the mass density ρ of the water, the dynamic viscosity μ of the water, and the roughness height ϵ of the concrete Prove that Q does not depend on ρ and μ separately, but only on the ratio μ/ρ What is the most general form of a dimensionally homogeneous equation for Q? If viscosity is negligible, how does Q vary with g?

25 How does the distance L from the leading edge of a smooth semi infinite thin plate to the point where the boundary layer becomes turbulent vary with the velocity V of the stream? How does L vary with the dynamic viscosity μ? With the mass density ρ?

26 The deflection d of a beam depends on the length L of the beam and the stiffness EI of the cross section, and it is directly proportional to the total load W on the beam How does d vary with L? With EI?

27 Liquid flows through an orifice of diameter D at the rate Q ft^3/sec. Derive the general form of the equation for the power loss caused by the orifice

28 A jet of liquid is directed vertically upward in the atmosphere List the variables that determine the maximum height to which the drops of liquid rise Make a dimensional analysis of the problem

CHAPTER 3

Systematic Calculation of Dimensionless Products

A part of the secret of analysis is the art of using notation well.

LEIBNIZ ON DETERMINANTS

Buckingham and a number of later writers on dimensional analysis have stated the rule that the number of dimensionless products in a complete set is equal to the total number of variables minus the number of fundamental dimensions in the problem. This is a convenient rule of thumb, but it is not infallible, as has been pointed out in Article 8. Bridgman[6] called attention to this fact in 1922. In 1946, Van Driest[7] stated the following rule, which may be rigorously proved:

> The number of dimensionless products in a complete set is equal to the total number of variables minus the maximum number of these variables that will not form a dimensionless product.

Another rule that is equivalent to Van Driest's rule will fit better into the developments of this chapter. Before discussing this rule, we shall recall some related algebraic principles. Proofs of the algebraic theorems may be found in books on advanced algebra.[8]

9. DETERMINANTS

An n'th order determinant is a square array of n^2 numbers, to which a value Δ is attached in a definite manner. A second order determinant is evaluated as follows:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

[6] P. W. Bridgman, *Dimensional Analysis*, Yale University Press, 1922.

[7] E. R. Van Driest, On Dimensional Analysis and the Presentation of Data in Fluid Flow Problems, *J. Applied Mechanics*, Vol. 13, no. 1, p. A-34, Mar. 1946.

[8] M. Bôcher, *Introduction to Higher Algebra*, Macmillan, New York, 1938.

Let a_{rc} be the number in the r'th row and the c'th column of an n'th order determinant An $(n-1)$ order determinant may be formed by crossing out the r th row and the c'th column of the given determinant The product of this $(n-1)$ order determinant with $(-1)^{r+c}$ is called the *cofactor* of the element a_{rc} With this definition, the following important theorem, known as *Laplace's development*, may be expressed:

> The sum of the products formed by multiplying all numbers in a row (or a column) of a determinant by their respective cofactors is the value of the determinant

Laplace s development enables us to reduce any determinant to determinants of lower order For example, expansion of the following determinant with respect to the fourth column yields

$$\Delta = \begin{vmatrix} 1 & 2 & -1 & 0 \\ 3 & 1 & -2 & 2 \\ 4 & -1 & 0 & 0 \\ 3 & -3 & 2 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & -1 \\ 4 & -1 & 0 \\ 3 & -3 & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & -2 \\ 4 & -1 & 0 \end{vmatrix}$$

In turn, Laplace's expansion of these third order determinants yields

$$\Delta = -(2)(4)\begin{vmatrix} 2 & -1 \\ -3 & 2 \end{vmatrix} + (2)(-1)\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} + (4)(4)\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix}$$
$$- (4)(-1)\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix}$$
$$= -(2)(4)(1) + (2)(-1)(5) + (4)(4)(-3) - (4)(-1)(1) = -62$$

Note the advantage in applying Laplace's development to rows or columns that contain zeros

10 Number of Dimensionless Products in a Complete Set

In order to utilize the algebraic approach to dimensional analysis, it is convenient to display the dimensions of the variables by a tabular arrangement Suppose, for example that the variables under consideration are velocity V, length L, force F, mass density ρ dynamic viscosity μ and acceleration of gravity g In the following table each column consists of the exponents in the dimensional expression for the corresponding variable For example, the dimension of g is $[M^0LT^{-2}]$, as is indicated by the last column in the table

	V	L	F	ρ	μ	g
M	0	0	1	1	1	0
L	1	1	1	−3	−1	1
T	−1	0	−2	0	−1	−2

In mathematics, a rectangular array of numbers is called a *matrix*. Accordingly, the above table is called the *dimensional matrix* of the variables.

A matrix is said to be "square" if the number of columns equals the number of rows. Any matrix contains square matrices that remain after certain rows or columns or both are crossed out of the original matrix. The determinants of these square matrices are called the "determinants of the original matrix." It occasionally happens that all determinants above a certain order are zero. Consequently, the following definition is employed in algebra:

If a matrix contains a nonzero determinant of order r, and if all determinants of order greater than r that the matrix contains have the value zero, the rank of the matrix is said to be r.

For example, the determinant formed from the last three columns in the above dimensional matrix is

$$\begin{vmatrix} 1 & 1 & 0 \\ -3 & -1 & 1 \\ 0 & -1 & -2 \end{vmatrix} = -3$$

Since this is a third order determinant that is different from zero, the rank of the dimensional matrix is 3.

In dimensional analysis, the importance of the concept of "rank of a matrix" stems from the following theorem which is proved in Chapter 4:

The number of dimensionless products in a complete set is equal to the total number of variables minus the rank of their dimensional matrix.

For example, since the rank of the above dimensional matrix is 3 and the number of variables is 6, the number of dimensionless products in a complete set is $6 - 3 = 3$. In fact, Reynolds' number, Froude's number, and the pressure coefficient are a complete set of dimensionless products of the given variables.

11. Linear Dependence

Consider the matrix,

$$\begin{pmatrix} 2 & 1 & 3 & 5 & 6 \\ 1 & -2 & 4 & 7 & 0 \\ 5 & 10 & 0 & -1 & 24 \end{pmatrix}$$

If the first row is multiplied by 4 and the second row is multiplied by -3 and the resulting two rows are added, the third row is obtained. Accordingly, the third row is said to be a *linear combination* of the other two rows. In general, if there exist constants corresponding to several rows of a matrix, such that the sum of the products of the several rows with their respective constants is another row of the matrix, that row is said to be a "linear combination" of the other rows

The rows of a matrix are said to be *linearly dependent* if there exists at least one row that is a linear combination of other rows. Otherwise, the rows of the matrix are said to be *linearly independent*. If a matrix has only two rows, linear dependence is equivalent to a proportion between the rows. Accordingly, linear dependence is a generalization of the concept of proportionality.

The following theorem concerning linear dependence is proved in algebra:

The rows of a matrix are linearly dependent if, and only if, the rank of the matrix is less than the number of rows.

In view of this theorem the rank of a matrix is not changed, if rows that are linear combinations of other rows are deleted. Frequently, it is apparent by inspection that one row of a dimensional matrix is proportional to another row or that it is a linear combination of several other rows. The computation of the rank of the dimensional matrix is greatly facilitated if those rows that are perceived to be linear combinations of other rows are deleted.

12 Example of Computation of Dimensionless Products

Let us consider the variables V, L, F, ρ, μ, g that were discussed in Article 10. Any* product π of these variables has the following form

$$\pi = V^{k_1} L^{k_2} F^{k_3} \rho^{k_4} \mu^{k_5} g^{k_6}$$

Whatever the values of the k's may be, the corresponding dimension of π is

$$[\pi] = [LT^{-1}]^{k_1}[L]^{k_2}[MLT^{-2}]^{k_3}[ML^{-3}]^{k_4}[ML^{-1}T^{-1}]^{k_5}[LT^{-2}]^{k_6}$$

This is apparent from the dimensions that are given in the dimensional matrix in Article 10. In accordance with the algebraic properties of exponents, the above dimensional expression for π may be written,

$$[\pi] = [M^{(k_3+k_4+k_5)} L^{(k_1+k_2+k_3-3k_4-k_5+k_6)} T^{(-k_1-2k_3-k_5-2k_6)}]$$

If π is required to be dimensionless, the exponents of M, L, and T must all be zero. Hence,

$$\begin{aligned} k_3 + k_4 + k_5 &= 0 \\ k_1 + k_2 + k_3 - 3k_4 - k_5 + k_6 &= 0 \qquad \text{(a)} \\ -k_1 - 2k_3 - k_5 - 2k_6 &= 0 \end{aligned}$$

* The use of the symbol π to denote a dimensionless product is conventional, it has no relation to the number 3.1416

Any solution of these equations is a set of exponents in a dimensionless product.

Observe that the coefficients in each equation are a row of numbers in the dimensional matrix. Therefore, the equations for the exponents of a dimensionless product can be written down directly by inspection of the dimensional matrix. This is invariably true. A proof of this principle may be obtained by applying the preceding reasoning to an arbitrary dimensional matrix.

Equation a consists of three equations in six unknowns. In mathematical terminology, the system of equations is "underdetermined." Such a system of equations possesses an infinite number of solutions. In the present case, any values may be assigned to three of the unknowns (say, k_1, k_2, and k_3), and the equations may be solved for the remaining three unknowns. Thus, the exponents of a dimensionless product are determined. For example, set $k_1 = -2$, $k_2 = -2$, $k_3 = 1$. Then Equation a yields $k_4 = -1$, $k_5 = 0$, $k_6 = 0$. Hence, a dimensionless product is $V^{-2}L^{-2}F\rho^{-1}$. This is the pressure coefficient. Similarly, if $k_1 = 1$, $k_2 = 1$, $k_3 = 0$, we obtain $k_4 = 1$, $k_5 = -1$, $k_6 = 0$, and the resulting dimensionless product is Reynolds' number, $VL\rho/\mu$. Finally, if $k_1 = 2$, $k_2 = -1$, $k_3 = 0$, we obtain $k_4 = 0$, $k_5 = 0$, $k_6 = -1$, and the resulting dimensionless product is Froude's number, V^2/Lg.

The foregoing procedure is quite arbitrary; any values might be chosen for k_1, k_2, and k_3. For example, set $k_1 = 10$, $k_2 = -5$, $k_3 = 8$. Then Equation a yields $k_4 = 8$, $k_5 = -16$, $k_6 = -5$. The resulting dimensionless product is

$$\pi = V^{10}L^{-5}F^8\rho^8\mu^{-16}g^{-5}$$

However, this is not really a new product, for it is determined by the pressure coefficient **P**, the Reynolds number **R**, and the Froude number **F**, as follows:

$$\pi = \mathbf{P}^8\mathbf{R}^{16}\mathbf{F}^5$$

A relationship of this form should have been anticipated, since **P**, **R**, and **F** are a complete set of dimensionless products of the variables. Regardless of the values that are assigned to k_1, k_2, and k_3, the resulting dimensionless product is a product of powers of **P**, **R**, and **F**. This fact and the condition that **P**, **R**, and **F** are independent of each other characterize them as a complete set of dimensionless products.

13. Theory of Homogeneous Linear Algebraic Equations

Suppose that we wish to form a dimensionless product of n variables. It has been shown in Article 12 that the exponents $(k_1, k_2, \cdots, k_n)$ in the dimensionless product are a solution of the linear algebraic equations,

$$a_1k_1 + a_2k_2 + \quad + a_nk_n = 0$$
$$b_1k_1 + b_2k_2 + \quad + b_nk_n = 0 \qquad \text{(a)}$$
$$. \quad . \ .$$

in which the coefficients a_i, b_i, etc , are the rows in the dimensional matrix Equations of this type are said to be *homogeneous* In this case, the word ' homogeneous" merely signifies that the terms on the right side of the equations are zero

If $(k'_1, k'_2, \quad, k'_n)$, $(k''_1, k''_2, \quad, k''_n)$, etc , are several solutions of Equation a, then $(k_1, k_2, \quad, k_n)$ is also a solution, in which

$$k_1 = Ak'_1 + Bk''_1 +$$
$$k_2 = Ak'_2 + Bk''_2 + \quad , \text{etc} \qquad \text{(b)}$$

where A, B, are any constants This may be proved by substituting Equation b in Equation a The solution $(k_1, k_2, \quad, k_n)$ is called a ' linear combination" of the solutions $(k'_1, k'_2, \quad k'_n)$, $(k''_1, k''_2, \quad, k''_n)$, etc

Solutions that are linear combinations of known solutions are, in a sense, trivial since an unlimited number of solutions of this type can be formed. Consequently, we are primarily interested in solutions that are linearly independent of each other The question then arises What is the maximum number of linearly independent solutions that Equation a possesses?" The following answer to this question is derived in the theory of linear algebra

Disregarding the trivial solution $k_i = 0$, Equation a possesses exactly $(n - r)$ linearly independent solutions, in which r is the rank of the matrix of the coefficients in Equation a A set of $(n - r)$ linearly independent solutions is called a fundamental system of solutions Any solution is a linear combination of the solutions in any fundamental system

By virtue of this theorem, all solutions of a set of linear homogeneous algebraic equations are effectively determined by any fundamental system of solutions

These algebraic principles have a direct bearing on dimensional analysis, since linearly independent solutions of Equation a furnish independent dimensionless products Consequently, any fundamental system of solutions furnishes a complete set of dimensionless products The following two articles illustrate a method for computing a fundamental system of solutions of any given set of linear homogeneous algebraic equations

14. Calculation of a Complete Set of Dimensionless Products

Consider the problem of computing dimensionless products of variables P, Q, R, S, T, U, V whose dimensional matrix is given below.

	1	2	3	4	5	6	7
	P	Q	R	S	T	U	V
M	2	−1	3	0	0	−2	1
L	1	0	−1	0	2	1	2
T	0	1	0	3	1	−1	2

The first step is to compute the rank of the matrix. Consequently, the determinant at the right side of the matrix is evaluated. This determinant is

$$\begin{vmatrix} 0 & -2 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 2 \end{vmatrix} = 1$$

Since this determinant is not zero, the rank of the matrix is $r = 3$. There is no theoretical reason for picking the determinant at the right side of the matrix; if the matrix contains *any* third order nonzero determinant, its rank is three. However, the procedure that is outlined below is based on the hypothesis that a nonzero determinant of order r occurs in the right-hand r columns of the matrix. The columns of the dimensional matrix must consequently be arranged so that this condition is satisfied. Occasionally, this may require a rearrangement of the columns after the rank has been computed. This does not change the rank.

Having computed the rank r, subtract this number from the number of variables in the problem. The difference is the number of dimensionless products in a complete set. Since, in the present example, the rank is three and the number of variables is seven, there are four dimensionless products in a complete set.

Now, write the homogeneous linear algebraic equations whose coefficients are the numbers in the rows of the dimensional matrix. In the present case, these equations are

$$\begin{aligned} 2k_1 - k_2 + 3k_3 - 2k_6 + k_7 &= 0 \\ k_1 - k_3 + 2k_5 + k_6 + 2k_7 &= 0 \\ k_2 + 3k_4 + k_5 - k_6 + 2k_7 &= 0 \end{aligned} \qquad \text{(a)}$$

Any values may be assigned to k_1, k_2, k_3, k_4, and Equation a may then be solved for k_5, k_6, k_7. It is consequently convenient to solve Equation a generally for k_5, k_6, k_7. The solution is accomplished readily by the elementary elimination procedure The result is

$$k_5 = -11k_1 + 9k_2 - 9k_3 + 15k_4$$

$$k_6 = 5k_1 - 4k_2 + 5k_3 - 6k_4 \qquad \text{(b)}$$

$$k_7 = 8k_1 - 7k_2 + 7k_3 - 12k_4$$

Let us now assign the values $k_1 = 1$ $k_2 = k_3 = k_4 = 0$ for the first solution Then, by Equation b $k_5 = -11$, $k_6 = 5$, $k_7 = 8$ Similarly, if $k_1 = 0$, $k_2 = 1$, $k_3 = k_4 = 0$, Equation b yields $k_5 = 9$ $k_6 = -4$, $k_7 = -7$ Likewise, Equation b yields solutions for the cases $k_1 = k_2 = 0$, $k_3 = 1$, $k_4 = 0$ and $k_1 = k_2 = k_3 = 0$ $k_4 = 1$ The solutions may be neatly arranged in the matrix form shown below

MATRIX OF SOLUTIONS

	k_1	k_2	k_3	k_4	k_5	k_6	k_7
	P	Q	R	S	T	U	V
π_1	1	0	0	0	−11	5	8
π_2	0	1	0	0	9	−4	−7
π_3	0	0	1	0	−9	5	7
π_4	0	0	0	1	15	−6	−12

Observe that the fifth sixth, and seventh columns in the matrix of solutions are merely the coefficients in the equations for k_5 k_6, and k_7 (Equation b) The first four columns of the matrix of solutions consist of zeros, except for the ones on the principal diagonal Consequently, the matrix of solutions can be written down immediately by inspection of Equation b It is apparent that this is always true

By considering the determinant at the left side of the matrix of solutions, we perceive that the rank of the matrix is invariably equal to the number of rows Consequently, the rows in the matrix of solutions are linearly independent Since the matrix of solutions contains $(n - r)$ rows, it constitutes a fundamental system of solutions

Each row in the matrix of solutions is a set of exponents in a dimensionless product Accordingly, in the present case, the following complete set of dimensionless products is obtained

$$\pi_1 = PT^{-11}U^5V^8 \qquad \pi_2 = QT^9U^{-4}V^{-7}$$

$$\pi_3 = RT^{-9}U^5V^7, \qquad \pi_4 = ST^{15}U^{-6}V^{-12}$$

Observe that the first variable P occurs only in π_1 the second variable Q occurs only in π_2 the third variable R occurs only in π_3 and the fourth variable S occurs only in π_4. This is an important characteristic of the method. It verifies the fact that the products are independent of each other. It is explained by the circumstance that the first part of the matrix of solutions consists of zeros, except for the terms on the principal diagonal.

15. Singular Dimensional Matrix

Exceptional cases arise in which the rank of the dimensional matrix is less than the number of rows that it contains. The dimensional matrix is then said to be *singular*. As an example, consider variables P, Q, R, S whose dimensional matrix is

	1	2	3	4
	P	Q	R	S
M	2	1	3	4
L	−1	6	−3	0
T	1	20	−3	8

All third order determinants in this matrix are zero; i.e.,

$$\begin{vmatrix} 2 & 1 & 3 \\ -1 & 6 & -3 \\ 1 & 20 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 4 \\ -1 & 6 & 0 \\ 1 & 20 & 8 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 4 \\ -1 & -3 & 0 \\ 1 & -3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 4 \\ 6 & -3 & 0 \\ 20 & -3 & 8 \end{vmatrix} = 0$$

However, the matrix contains a second order determinant that is different from zero. For example, the second order determinant in the upper right-hand corner is

$$\begin{vmatrix} 3 & 4 \\ -3 & 0 \end{vmatrix} = 12$$

Therefore, the rank of the matrix is $r = 2$. The arrangement of the matrix is satisfactory, since a nonzero determinant of order r occurs in the right-hand r columns. Since there are four columns in the matrix and the rank is two, the number of dimensionless products in a complete set is $4 - 2 = 2$.

In general, it is not necessary to consider all rows in the dimensional matrix, if the rank r is less than the number of rows. Rather, it suffices to consider r rows whose rank is r. Since, in the present case, $r = 2$, and since the first two rows of the dimensional matrix are themselves a matrix of rank two, the third row may be discarded altogether. The homogeneous linear algebraic equations corresponding to the first two rows are

$$\begin{aligned} 2k_1 + k_2 + 3k_3 + 4k_4 &= 0 \\ -k_1 + 6k_2 - 3k_3 &= 0 \end{aligned} \qquad \text{(a)}$$

Two independent solutions of these equations are required, since there are two dimensionless products in a complete set The following matrix of solutions is obtained

	k_1 P	k_2 Q	k_3 R	k_4 S
π_1	1	0	$-\frac{1}{3}$	$-\frac{1}{4}$
π_2	0	1	2	$-\frac{7}{4}$

Note that each row of this matrix is a solution of Equation a Also note that, if the equation corresponding to the third row of the dimensional matrix is written (namely $k_1 + 20k_2 - 3k_3 + 8k_4 = 0$), each row in the matrix of solutions is a solution of this equation This follows from the fact that the third equation is a linear combination of the two Equations a in fact, it is obtained by multiplying the first equation by 2 and the second equation by 3 and adding the resulting equations In general, the equations for the exponents of dimensionless products are linearly dependent if the rank of the dimensional matrix is less than the number of rows For this reason it is generally permissible to delete all but r rows of the dimensional matrix in which r is the rank of the matrix However, it is essential *that the matrix that remains after the rows are deleted shall still have rank* r

The complete set of dimensionless products corresponding to the above matrix of solutions is

$$\pi_1 = PR^{-1/3}S^{-1/4} \qquad \pi_2 = QR^2S^{-7/4}$$

Since a power of a dimensionless product is still a dimensionless product it is permissible to raise a dimensionless product to a power that eliminates fractional exponents Consequently, the products that have been found may be replaced by

$$\pi_1 = P^{12}R^{-4}S^{-3}, \qquad \pi_2 = Q^4R^8S^{-7}$$

16 Arrangement of Variables

There are infinitely many different complete sets of dimensionless products that can be formed from a given set of variables Insofar as Buckingham's theorem is concerned any complete set of dimensionless products is admissible However, Buckingham[3] has demonstrated, with the aid of a well-chosen example, that some sets of products are more useful in practice than others and that certain transformations of the π's may bring an equation $\pi = f(\pi_1, \pi_2, \quad, \pi_p)$ into a more tractable form Rather than pursue a theory of transformations of dimensionless products, however, it is better to inquire "How may a complete set of dimensionless products be most advantageously selected at the outset?" The answer to this question does not depend entirely on arbitrary definitions for the experimenter de sires that any one of the independent dimensionless variables $\pi_1, \pi_2, \cdots, \pi_p$ be susceptible to control by experimental techniques while the others are held constant. This is sometimes too much to demand, because often only a few of the original variables can be experimentally regulated. For example, the velocity of fluid in a pipe can be regulated by a valve. On the other hand, the acceleration of gravity is a variable that we cannot change.

Buckingham has pointed out that we obtain the maximum amount of experimental control over the dimensionless variables if the original variables that can be regulated each occur in only one dimensionless product. For example, if a velocity V is easily varied experimentally, then V should occur in only one of the independent dimensionless variables. That dimensionless variable can then be regulated by varying V. Likewise, if a pressure p can be easily varied without affecting V, then p should occur in only one of the independent dimensionless variables, but not in the same one as V.

The dependent variable of the problem must also be considered. It is desired to know how this variable depends on the other variables. The dependent variable consequently should not occur in more than one dimensionless product. This product will be called the "dependent dimensionless variable."

Since the first $(n - r)$ variables in the dimensional matrix each occur in only one dimensionless product, the preceding conditions will be realized, as nearly as possible, if the following rule is observed:

In the dimensional matrix, let the first variable be the dependent variable. Let the second variable be that which is easiest to regulate experimentally. Let the third variable be that which is next easiest to regulate experimentally, and so on.

In exceptional cases, this arrangement may lead to an impasse, because the dimensional matrix does not contain a nonzero determinant of order r in the right-hand r columns. The variables in the dimensional matrix should then be rearranged without altering the recommended arrangement more than necessary.

17. Transformations of Dimensionless Products

In the preceding article, attention was called to Buckingham's proposal for transforming dimensionless products to achieve greater experimental control of the variables. Occasionally transformations are desirable for other reasons. For example, after a dimensional analysis of a problem has been performed, it may be decided that a certain variable that was introduced in the dimensional matrix has a negligible influence on the phenomenon Then, if this variable occurs in only one of the independent dimensionless variables, that dimensionless variable may be discarded However, if the variable that is to be neglected occurs in more than one dimensionless product, it is obviously incorrect to discard all dimensionless products in which it occurs It is then necessary to change to another complete set of dimensionless products Also, this is sometimes desirable, in order to obtain standard products such as Reynolds' number and Froude's number

Various complete sets of dimensionless products can be formed from a given complete set For example, if, in the dimensional analysis of the drag on a ship (Example 4), the variables are written in the order (F, V, L, μ, ρ, g) (which agrees with the order recommended in Article 16), the following result is obtained by the method of Article 14

$$f(\pi_1, \pi_2, \pi_3) = 0$$

in which

$$\pi_1 = \frac{\rho F}{\mu^2}, \qquad \pi_2 = V\sqrt[3]{\frac{\rho}{\mu g}}$$

$$\pi_3 = L\sqrt[3]{\frac{\rho^2 g}{\mu^2}}$$

Now, suppose that it is desired to neglect the viscosity μ Obviously, the products that contain μ cannot be discarded since μ occurs in every product However, from the products (π_1 π_2, π_3), another complete set may be obtained as follows

$$\mathbf{P} = \frac{\pi_1}{\pi_2^2\pi_3^2} = \frac{F}{\rho V^2 L^2}$$

$$\mathbf{R} = \pi_2\pi_3 = \frac{VL\rho}{\mu}, \qquad \mathbf{F} = \frac{\pi_2^2}{\pi_3} = \frac{V^2}{Lg}$$

Consequently, the equation $f(\pi_1\ \pi_2, \pi_3) = 0$ may be written $f(\mathbf{P}, \mathbf{R}, \mathbf{F}) = 0$ or $\mathbf{P} = f(\mathbf{R}\ \mathbf{F})$ which agrees with the result that was derived in Example 4 Now, if μ is considered to be negligible the term **R** may be discarded

The justification for replacing $f(\pi_1, \pi_2\ \pi_3)$ by $f(\mathbf{P}, \mathbf{R}, \mathbf{F})$ is that the relationship among π_1, π_2, and π_3 is unknown all that is known is that a relationship exists *A relationship among* π_1, π_2 *and* π_3 *implies a relationship among* **P**, **R**, and **F**, since the variables π_1, π_2 and π_3 are determined by **P**, **R**, and **F**, and vice versa This follows from the fact that either set of dimensionless variables is complete

When a transformation of dimensionless products is performed, it is necessary to ascertain that there are as many new products as original products and that the new products are independent of each other. Otherwise, the new products do not form a complete set.

EXAMPLE 8. EFFECT OF TEMPERATURE ON THE VISCOSITY OF A GAS*

In many applications of the kinetic theory of gases, it is unnecessary to consider the details of the structure of a molecule. A molecule may be conceived as a tiny ball of fog. Forces of attraction between molecules may be neglected. However, when two molecules come so close together that they begin to interpenetrate each other, they exert a strong repulsive force. This force is believed to be proportional to an inverse power of the distance; i.e. $F = Kr^{-n}$, in which r is the distance between the centers of the molecules and n is a numerical exponent that is probably greater than five. The coefficient K is a characteristic property of the molecules.

Rayleigh based his analysis on the law that the viscosity of a gas does not depend on the density. This principle was deduced by Maxwell from molecular considerations. It has been found to be fairly accurate for pressures in the range 0.02 atm to 1 atm. For pressures exceeding a few atmospheres, it is usually not tenable, partly because intermolecular attractions come into play in dense gases.

If the viscosity of a gas does not depend on the density, it does not depend on molecular characteristics that are related to the density: e.g., the number of molecules per unit volume, or the mean free path of a molecule. Consequently, the viscosity μ must be determined by the mass m of a molecule, the mean velocity V of a molecule, and the coefficient of repulsion K. This is indicated by the equation,

$$f(\mu, K, m, V) = 0$$

The dimensional matrix of the variables is

	μ	K	m	V
M	1	1	1	0
L	−1	$n+1$	0	1
T	−1	−2	0	−1

The rank of this matrix is three. Therefore, there is only one dimensionless product in a complete set. The equations corresponding to the dimensional matrix are

$$k_1 + k_2 + k_3 = 0$$

$$-k_1 + (n+1)k_2 + k_4 = 0$$

$$-k_1 - 2k_2 - k_4 = 0$$

* The treatment of this problem by dimensional analysis is due to Rayleigh. (Reference 4)

Setting $k_1 = 1$ and solving the equations, we get

$$k_1 = 1, \quad k_2 = \frac{2}{n-1}, \quad k_3 = -\frac{n+1}{n-1}, \quad k_4 = -\frac{n+3}{n-1}$$

Consequently, a complete set of dimensionless products consists of the single product,

$$\pi = \mu K^{2/(n-1)} m^{-(n+1)/(n-1)} V^{-(n+3)/(n-1)}$$

Buckingham's theorem yields $f(\pi) = 0$, whence $\pi = \alpha$, a constant. It follows that

$$\mu = \alpha m^{(n+1)/(n-1)} V^{(n+3)/(n-1)} K^{-2/(n-1)}$$

It is shown in the kinetic theory of gases that the absolute temperature θ of a gas is proportional to the kinetic energy $\frac{1}{2}mV^2$ of a molecule. Consequently the preceding equation may be expressed

$$\mu = \beta m^{1/2} K^{-2/(n-1)} \theta^s, \quad \text{where} \quad s = \frac{1}{2} + \frac{2}{n-1} \tag{a}$$

The factor β is a constant.

For a given gas m and K are constants. Consequently, since $n > 1$, Equation a shows that the viscosity of a gas increases with the temperature. Since, by Equation a, the viscosity is proportional to a power of θ, the relationship between μ and θ is represented by a straight line on logarithmic graph paper.

Equation a provides information about the forces of repulsion between molecules. For $n = 5$, μ is proportional to the first power of the absolute temperature, and, for $n = \infty$, μ is proportional to the square root of the absolute temperature. Rayleigh found experimentally that $s = 0.754$ for air, $s = 0.782$ for oxygen, $s = 0.681$ for hydrogen, and $s = 0.815$ for argon. These results show that n is in the range 7 to 12 for common gases.

EXAMPLE 9 FRICTION OF A JOURNAL BEARING

Dimensional analysis has received numerous applications in the theory of lubrication. A simple, yet important, application is illustrated in this example.

Oil is delivered to a journal bearing through an oil hole, and it flows out at the ends of the bearing. In a bearing of this type, the journal is supported by a thick film of oil. The frictional resistance of a bearing is commonly designated by a dimensionless friction coefficient f, which is defined by $f = 2T/WD$, in which T is the resisting torque of the bearing, W is the load on the bearing, and D is the diameter of the journal. The load on the bearing is frequently designated by the average bearing pressure P, which is defined by $P = W/LD$, in which L is the length of the bearing.

When the journal rotates at constant angular speed N, a condition of thermal equilibrium is established, in which heat is conducted and convected away as fast as it is generated. The viscosity μ of the oil at the equilibrium temperature is naturally one of the variables that determines the friction coefficient f. Also, the clearance C of the bearing (difference between the diameters of the bearing and the journal) is a significant variable.

If a shaft is supported by a single bearing, the loads that are applied to the shaft ordinarily exert a moment M on the bearing about an axis perpendicular to the shaft. For the sake of generality, a moment of this type is taken into account in the analysis of the problem.

The friction coefficient f is a function of the variables P, M, L, D, C, μ, N. Since L/D and C/D are seen to be dimensionless products, the variables L and C may be tentatively disregarded. Then the dimensional matrix is

	1	2	3	4	5
	P	M	D	μ	N
M	1	1	0	1	0
L	−1	2	1	−1	0
T	−2	−2	0	−1	−1

The rank of this matrix is three. Accordingly, the matrix furnishes two independent dimensionless products. The products $\mu N/P$ and M/PD^3 are readily discovered by inspection. Consequently, by Buckingham's theorem, the general form of the equation for the friction coefficient is

$$f = f\left(\frac{\mu N}{P}, \frac{M}{PD^3}, \frac{L}{D}, \frac{C}{D}\right)$$

Thus, the number of independent variables is reduced from seven to four. This is a great advantage in the testing of bearings and lubricants. Frequently, the moment M does not exist, in which case there are only three independent variables.

Problems

1. Prove that (**R**, **P**, **F**, **M**, **W**) is a complete set of dimensionless products of the variables (F, L, V, ρ, μ, g, c, σ).

2. In the example in Article 12, set $k_1 = a$, $k_2 = b$, $k_3 = c$. Then what are the expressions for k_4, k_5, and k_6? What are the expressions for r, s, and t in the following equation?

$$V^{k_1}L^{k_2}F^{k_3}\rho^{k_4}\mu^{k_5}g^{k_6} = \mathbf{P}^r\mathbf{R}^s\mathbf{F}^t$$

3. Make up an example of a dimensional matrix, in which the right-hand r columns do not contain a nonzero determinant of order r. What difficulty is encountered in forming the matrix of solutions in this case? (r = rank of the dimensional matrix)

4 *Calculate a complete set of dimensionless products of the following variables* Volume Q, acceleration A, velocity V, power P, momentum M, angular velocity N

Determine the ranks of the following dimensional matrices and the numbers of dimensionless products in complete sets Calculate complete sets of dimensionless products Eliminate fractional exponents

5

	A	B	C	D	F	F
M	1	1	−1	0	0	−2
L	3	2	1	−1	−4	0
T	−1	−2	2	0	3	1

6

	A	B	C	D	E	F	G	H
M	1	1	0	−2	0	1	−1	2
L	2	2	0	1	4	−2	−3	5
T	−3	2	0	−1	−4	3	1	4

7

	A	B	C	D
M	−7	−2	−3	14
L	−2	−4	3	1
T	−1	2	−3	4

8

	A	B	C	D	E	F	G
M	0	0	1	2	−1	3	−2
L	2	6	−3	0	1	0	1
T	0	1	−1	−5	−2	2	1
Q	1	2	0	0	0	−1	−3

9

	A	B	C	D	F	F
M	3	1	0	2	−1	−1
L	0	2	1	1	0	1
T	3	3	1	3	−1	0
Q	0	4	2	2	0	2

10

	A	B	C	D
M	1	−1	2	0
L	−3	0	1	−2
T	−1	−2	5	−2
Θ	4	−1	1	2

11 The speed V of the wind that creates white caps on the surface of the ocean depends on the mass densities ρ_w and ρ_a of water and air the viscosities μ_w and μ_a of water and air, and the acceleration of gravity g What is the most general form of a dimensionally homogeneous equation for V?

12 The speed V of the wind that creates ripples on the surface of shallow water depends on the mass densities ρ_w and ρ_a of water and air the viscosities μ_w and μ_a of water and air, the depth h of the water and the surface tension σ of water What is the most *general form of a dimensionally homogeneous equation for V?*

13 A liquid is poured at a constant rate Q (ft³/sec) into a spinning conical cup, and it is subjected to radiation to kill bacteria as it flows up the wall of the cup under centrifugal action The effectiveness of the radiation depends on the thickness of the layer of fluid on the wall This is determined by the rate of flow Q, the angular velocity n of

the cup, the mass density ρ of the fluid, the dynamic coefficient of viscosity μ of the fluid, the acceleration of gravity g, the height H of the cup, and the angle α of the cone. Determine the most general form of a dimensionally homogeneous equation for the average thickness h of the layer of fluid.

14. The maximum pitching moment M that is experienced by a flying boat while landing is a function of the following variables:

α, the angle that the flight path makes with the horizontal.
β, the angle that defines the attitude of the ship.
V, the landing speed.
m, the mass of the ship.
R, the radius of gyration of the ship with respect to the axis of pitching.
L, a length that specifies the size of the hull.
ρ, the mass density of the water.
g, the acceleration of gravity.

Make a dimensional analysis of the problem, suitable for plotting data from landing tests.

15. Prove that the velocity of ripples on the surface of a liquid is proportional to the square root of the surface tension, inversely proportional to the square root of the mass density, and inversely proportional to the square root of the wave length. Neglect effects of gravity, viscosity, and wave amplitude.

16. If a smooth ball falls through a homogeneous fluid, it eventually acquires a "terminal velocity" at which the acceleration ceases, since the weight of the ball is balanced by the buoyant force and the resistance of the fluid. If the terminal velocity is not so great that the compressibility of the fluid is significant, it depends on the viscosity μ of the fluid, the mass density ρ of the fluid, the diameter D of the ball, and the weight W' of the ball in the fluid. (W' is the true weight minus the buoyant force.) Determine dimensionless coordinates of a curve that gives complete information concerning terminal velocities of smooth balls falling through fluids, neglecting compressibility. Explain how to plot this curve from observations of steel balls falling in water.

17. Prove that the frequency of any mode of vibration of a drop of liquid, under the action of its surface tension, is proportional to the square root of the surface tension, inversely proportional to the square root of the mass density, and inversely proportional to the 3/2 power of the diameter. (Rayleigh)

18. Assume that the rate of flow Q (ft^3/sec) over a rectangular weir is independent of the viscosity and that it is proportional to the width of the weir. Then prove that Q is proportional to the 3/2 power of the height of the water level above the edge of the weir.

19. Neglecting viscosity, prove that the rate of flow Q (ft^3/sec) over a triangular weir is proportional to the 5/2 power of the height of the water level above the notch of the weir.

20. List the variables that determine the maximum diameter of a drop of liquid that will not disintegrate while falling through a gas (e.g. a raindrop falling in air). Determine dimensionless coordinates of a single curve that completely defines the relationship.

21. List the variables that determine the terminal velocity of a falling raindrop. Determine the most general dimensionally homogeneous form of an equation that expresses this relationship.

22. List the variables that determine the amplitude of oscillation of a flexible elastic rod that is forced to vibrate by the wind. Derive the most general dimensionally homogeneous form of an equation that expresses this relationship.

23. An airplane is flying through a rainstorm. Assuming that all raindrops have the same diameter and that the shape of the nose of the fuselage is given, list the variables

that determine the number of raindrops that strike the windshield per second Make a dimensional analysis of the problem

24 A uniform wind in a desert lifts sand into the air Assuming that all sand grains have the same diameter, list the variables that determine the weight of sand in the air, per unit area of land surface Make a dimensional analysis of the problem

25 An airplane is warming up its engine on the ground List the variables that determine the intensity U of sound energy (energy per unit volume) from the propeller at a distance L ahead of the airplane Make a dimensional analysis of the problem

26 If a drop of liquid falls into a pool, a small column of liquid splashes out of the pool List the variables that determine the height h of the column Make a dimensional analysis of the problem

27 Make up an example of a 3 rowed matrix whose rows are linearly dependent Verify that the rank of the matrix is less than the number of rows

CHAPTER 4

Algebraic Theory of Dimensional Analysis

All things are numbers. PYTHAGORAS

Dimensional analysis is so closely knitted with physical concepts that abstract statements of its theorems are not immediately apparent. It is possible, however, to strip the physical ideas from dimensional analysis, and there remains a set of algebraic theorems concerning a class of functions that is characterized by a generalized type of homogeneity. These algebraic theorems, which culminate in Buckingham's theorem, are developed in this chapter.[9]

For the sake of simplicity, only the three fundamental dimensions $[M]$, $[L]$, and $[T]$ will be considered. It should be apparent that the physical significance of the fundamental dimensions is entirely irrelevant and that the conclusions are essentially unchanged if there are n fundamental dimensions $[M]$, $[L]$, $[T]$, $[\Theta]$, $[Q]$, $\cdots$, $[\Psi]$.

18. General Formula for Transforming Units of Measurement

The number that specifies the distance from an origin on an axis to a point P on the axis is called the "coordinate" of the point P. In graphical representations, the idea of coordinates is extended to other scalar entities than distance; e.g., we may have a time axis, a temperature axis, etc. Consequently, it is natural to refer to the variable which specifies the magnitude of any scalar entity as the "coordinate of the magnitude."

The general method by which the coordinate of a magnitude is transformed when the basic units of measurement are changed has been explained in Article 4. It will now be shown that this method may be expressed by a formula. The desired formula is the solution of the following problem:

> The dimension of an entity is $[M^aL^bT^c]$, and the coordinate of the magnitude of the entity is x, when mass, length, and time are meas-

[9] Through the courtesy of the Franklin Institute, the material in this chapter is adapted from the author's paper, A Summary of Dimensional Analysis, *J. Franklin Inst.*, Vol. 242, no. 6, p. 459, Dec. 1946.

ured in certain units, called the "original units" New units are introduced, such that

$$1 \text{ original mass unit} = A \text{ new mass units}$$

$$1 \text{ original length unit} = B \text{ new length units}$$

$$1 \text{ original time unit} = C \text{ new time units}$$

In terms of the new units, the coordinate of the magnitude is $\bar{x}$ What is the relationship between x and $\bar{x}$?

This problem is directly solvable by the method of Article 4 Denote the original units of mass length, and time by (OM), (OL), and (OT), and the new units by (NM) (NL) and (NT), respectively* Then,

$$x[(OM)^a(OL)^b(OT)^c] = x[A^a(NM)^aB^b(NL)^bC^c(NT)^c] = \bar{x}[(NM)^a(NL)^b(NT)^c]$$

It follows that

$$\bar{x} = xA^aB^bC^c \tag{4}$$

Equation 4 is an algebraic formula for the transformation that the coordinate x of a magnitude undergoes when the units of mass, length, and time are subjected to any changes

For example to express an acceleration of 900 ft/min² in the unit 'in/sec²,' set $x = 900$, $B = 12$ $C = 60$ Then by Equation 4,

$$\bar{x} = 900A^0(12)(60)^{-2} = 3 \text{ in/sec}^2$$

19 Mathematical Definition of Dimensional Homogeneity

Let y be a function of n variables i e, $y = f(x_1, x_2, \ldots, x_n)$ The symbol f may be regarded as an operator that is applied to the independent variables x_1 x_2, …, x_n to yield the proper value of the dependent variable y If the basic units of measurement are subjected to changes, the variables take new values, $\bar{x}_1, \bar{x}_2$ … $\bar{x}_n$ $\bar{y}$ By definition (cf Article 5), the equation is dimensionally homogeneous if and only if,

$$\bar{y} = f(\bar{x}_1\ \bar{x}_2, \ldots, \bar{x}_n) \tag{a}$$

in which f is the same operator as before This condition is mathematically expressed by the statement that the equation is 'invariant" under the group of transformations that is generated by all possible changes of the units of mass length and time Now, this group of transformations is defined by Equation 4 in which A, B and C are arbitrary positive constants

* These symbols take the place of words such as "slug" 'foot' and second"

The dimensions of the variables will be designated by the following dimensional matrix:

$$\begin{array}{c|cccc} & y & x_1 & x_2 \cdots & x_n \\ \hline M & a & a_1 & a_2 \cdots & a_n \\ L & b & b_1 & b_2 \cdots & b_n \\ T & c & c_1 & c_2 \cdots & c_n \end{array} \tag{5}$$

Then Equation 4 yields

$$\begin{aligned} \bar{y} &= yA^aB^bC^c = yK \\ \bar{x}_1 &= x_1A^{a_1}B^{b_1}C^{c_1} = x_1K_1 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \bar{x}_n &= x_nA^{a_n}B^{b_n}C^{c_n} = x_nK_n \end{aligned} \tag{6}$$

in which the K's are defined by

$$\begin{aligned} K &= A^aB^bC^c \\ K_i &= A^{a_i}B^{b_i}C^{c_i}, \qquad i = 1, 2, \cdots, n \end{aligned} \tag{7}$$

Substitution of Equation 6 in Equation a yields

$$Kf(x_1, x_2, \cdots, x_n) = f(K_1x_1, K_2x_2, \cdots, K_nx_n) \tag{8}$$

Thus, the following principle is established:

Theorem 1. The function $f(x_1, x_2, \cdots, x_n)$ is dimensionally homogeneous if, and only if, Equation 8 is an identity in the variables $(x_1, x_2, \cdots, x_n, A, B, C)$.

Note that the K's are all determined by the three numbers, A, B, C.

For example, the drag force on a spherical body in a stream of incompressible fluid (Article 6) is given by an equation of the type,

$$F = f(V, D, \rho, \mu)$$

By Equation 8, the condition of dimensional homogeneity is

$$KF = f(K_1V, K_2D, K_3\rho, K_4\mu) \tag{b}$$

The dimensional matrix is

$$\begin{array}{c|ccccc} & F & V & D & \rho & \mu \\ \hline M & 1 & 0 & 0 & 1 & 1 \\ L & 1 & 1 & 1 & -3 & -1 \\ T & -2 & -1 & 0 & 0 & -1 \end{array}$$

Hence, by Equation 7,

$$K = ABC^{-2}, \quad K_1 = BC^{-1}, \quad K_2 = B, \quad K_3 = AB^{-3}, \quad K_4 = AB^{-1}C^{-1}$$

Equation b must be an identity in the variables A, B, C. It may be readily seen that this condition is automatically satisfied by the equation,

$$F = \rho V^2 D^2 f\left(\frac{VD\rho}{\mu}\right)$$

20 Dimensional Homogeneity of a Sum

Theorem 1 serves to verify the theorem that a sum of terms is dimensionally homogeneous if, and only if, all terms in the sum have the same dimension as the sum For let f be the sum of the x's, i e ,

$$y = f(x_1, x_2, \cdots, x_n) = x_1 + x_2 + \cdots + x_n$$

Then Equation 8 becomes

$$K(x_1 + x_2 + \cdots + x_n) = K_1 x_1 + K_2 x_2 + \cdots + K_n x_n$$

Since this is an identity in the x's,

$$K = K_1 = K_2 = \cdots = K_n$$

It follows, from Equation 7,

$$a = a_1 = a_2 = \cdots = a_n$$

$$b = b_1 = b_2 = \cdots = b_n$$

$$c = c_1 = c_2 = \cdots = c_n$$

This means that the variables y $x_1, x_2 \cdots, x_n$ all have the same dimension Accordingly this is a necessary condition for dimensional homogeneity of the sum By reversing the argument, it may be seen that the condition is also sufficient

In most presentations of dimensional analysis, the condition of dimensional homogeneity of a sum of terms is adopted as a general definition, i e , an equation is said to be dimensionally homogeneous if and only if, all its terms have the same dimension From a mathematical viewpoint, this definition is unsatisfactory, since the concept of "terms" does not enter into the definition of a function In mathematics, y is said to be a function of x if, to each value of x, there corresponds a value of y For example, a function may be defined by a graph Here, the idea of terms in an equation is not involved However, the concept of dimensional homogeneity is not ruled out in this case if Theorem 1 is adopted as a definition According to this definition, a function that is defined by a graph is dimensionally homogeneous if and only if, the curve remains unchanged when the basic units of measurement are changed in any way For example, the area A of a square whose side is L is represented by a parabolic graph with ordinate A and abscissa L This graph is valid irrespective of the unit of length. Therefore, the relationship is dimensionally homogeneous. On the other hand, the stress-strain relationship of a material is not dimensionally homogeneous, since the form of the ordinary tension stress-strain curve depends on the unit of stress.

21. Dimensional Homogeneity of a Product

It has been shown in the preceding chapters that expressions of the type

$$y = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \tag{a}$$

play an important part in dimensional analysis. Such expressions are briefly called "products."

If the dimensions of the variables $(y, x_1, x_2, \cdots, x_n)$ are represented by Equation 5, the exponents of the product y satisfy the following conditions:

Theorem 2. The product y is dimensionally homogeneous if, and only if, the exponents $(k_1, k_2, \cdots, k_n)$ are a solution of the linear equations,

$$\begin{aligned} a_1k_1 + a_2k_2 + \cdots + a_nk_n &= a \\ b_1k_1 + b_2k_2 + \cdots + b_nk_n &= b \\ c_1k_1 + c_2k_2 + \cdots + c_nk_n &= c \end{aligned} \tag{9}$$

To show that this condition is necessary, let y be a dimensionally homogeneous product with dimensional exponents (a, b, c). Then the function satisfies Equation 8. In view of Equation a, this relationship takes the more special form,

$$K x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} = K_1^{k_1} x_1^{k_1} K_2^{k_2} x_2^{k_2} \cdots K_n^{k_n} x_n^{k_n}$$

It follows that $$K = K_1^{k_1} K_2^{k_2} \cdots K_n^{k_n}$$

By virtue of Equation 7 this yields Equation 9. Thus, it is demonstrated that Equation 9 is a necessary condition. The proof of sufficiency is obtained by reversing the preceding proof.

Observe that, if y is dimensionless, Equation 9 is the set of linear homogeneous equations whose coefficients are the numbers in the rows of the dimensional matrix.

22. Complete Sets of Dimensionless Products

Let the following expressions be dimensionless products:

$$\begin{aligned} \pi_1 &= x_1^{k'_1} \quad x_2^{k'_2} \cdots x_n^{k'_n} \\ \pi_2 &= x_1^{k''_1} \quad x_2^{k''_2} \cdots x_n^{k''_n} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \pi_p &= x_1^{k_1^p} \quad x_2^{k_2^p} \cdots x_n^{k_n^p} \end{aligned}$$

The exponents of the x's in these products are displayed by the following matrix

$$\begin{array}{c|ccc} & x_1 & x_2 & x_n \\ \hline \pi_1 & k'_1 & k'_2 & k'_n \\ \pi_2 & k''_1 & k''_2 & k''_n \\ & & & \\ \pi_p & k_1^p & k_2^p & k_n^p \end{array} \tag{a}$$

In accordance with the definition in Chapter 2, the products $\pi_1, \pi_2, \cdots, \pi_p$ are said to be *independent*, if there are no constants $h_1, h_2, \cdots, h_p$, other than $h_1 = h_2 = \cdots = h_p = 0$, such that

$$\pi_1^{h_1}\pi_2^{h_2}\cdots\pi_p^{h_p} \equiv 1$$

The following theorem concerning independent dimensionless products will now be proved

Theorem 3 A necessary and sufficient condition that the products $\pi_1, \pi_2, \cdots, \pi_p$ be independent is that the rows in the matrix of exponents (Equation a) be linearly independent

We shall prove this theorem by assuming conditions contrary to the conclusion and showing that this leads to a contradiction of the hypothesis This shows that assumptions contrary to the theorem are inconsistent Hence, the theorem must be true

(a) *Proof that the Condition Is Necessary* Let the products be independent, and assume that the rows in the matrix of exponents are linearly dependent Then by the definition of linear dependence (Article 11), there exist constants $(h_1, h_2, \cdots, h_p)$ (not all zero), such that

$$h_1k'_i + h_2k''_i + \cdots + h_pk_i^p = 0, \qquad i = 1, 2, \cdots, n \tag{b}$$

Now

$$\pi_1^{h_1}\pi_2^{h_2}\cdots\pi_p^{h_p} = x_1^{(h_1k'_1+\cdots+h_pk_1^p)}x_2^{(h_1k'_2+\cdots+h_pk_2^p)}\cdots$$

It follows, from Equation b,

$$\pi_1^{h_1}\pi_2^{h_2}\cdots\pi_p^{h_p} = x_1^0x_2^0\cdots x_n^0 = 1$$

This is contrary to the hypothesis that the products are independent Thus it is proved that when the products are independent, the rows in the matrix of exponents are linearly independent

(b) *Proof that the Condition Is Sufficient* Let the rows in the matrix of exponents be linearly independent, and assume that the dimensionless products are dependent, i e that there exist constants $(h_1, h_2, \cdots, h_p)$ (not all zero), such that

$$\pi_1^{h_1}\pi_2^{h_2}\cdots\pi_p^{h_p} \equiv 1$$

Then

$$x_1^{(h_1k'_1+\cdots+h_pk_1^p)}x_2^{(h_1k'_2+\cdots+h_pk_2^p)}\cdots x_n^{(h_1k'_n+\cdots+h_pk_n^p)} = 1$$

Since this is an identity in the x's, the exponents vanish. But this is contrary to the hypothesis that the rows in the matrix of exponents are linearly independent. Thus, it is proved that, when the rows in the matrix of exponents are linearly independent, the dimensionless products are independent.

Turning now to the question of computation of dimensionless products, we observe that a product is dimensionless if, and only if, the exponents k_i are a solution of the equations,

$$\begin{aligned} a_1k_1 + a_2k_2 + \cdots + a_nk_n &= 0 \\ b_1k_1 + b_2k_2 + \cdots + b_nk_n &= 0 \\ c_1k_1 + c_2k_2 + \cdots + c_nk_n &= 0 \end{aligned} \tag{10}$$

in which the coefficients are the rows in the dimensional matrix of the x's. This follows directly from Theorem 2. Any fundamental system of solutions of Equation 10 furnishes $(n - r)$ linearly independent sets of exponents $k'_i, k''_i, \cdots, k_i^{(n-r)}$, $(i = 1, 2, \cdots, n)$, in which r is the rank of the dimensional matrix (i.e., the matrix of coefficients of Equation 10). By Theorem 3, these exponents, being linearly independent, yield independent dimensionless products. Furthermore, it is impossible to have more than $(n - r)$ independent dimensionless products, since Equation 10 possesses no more than $(n - r)$ linearly independent solutions. Accordingly, the following important theorem is obtained:

Theorem 4. Any fundamental system of solutions of Equation 10 furnishes exponents of a complete set of dimensionless products of the variables $x_1, x_2, \cdots, x_n$. Conversely, the exponents of a complete set of dimensionless products of the variables $x_1, x_2, \cdots, x_n$ are a fundamental system of solutions of Equation 10.

In view of the properties of a fundamental system of solutions, the following corollary to this theorem is obvious:

Theorem 5. The number of products in a complete set of dimensionless products of the variables $x_1, x_2, \cdots, x_n$ is $n - r$, in which r is the rank of the dimensional matrix of the variables.

23. Products That Are Not Dimensionless

Suppose that, in the dimensional matrix (Equation 5), the constants a, b, c are not all zero. Under what conditions does there exist a product of type $y = x_1^{k_1}x_2^{k_2} \cdots x_n^{k_n}$? The answer to this question is given by the following theorem:

Theorem 6. If y is not dimensionless, a product of the form $y = x_1^{k_1}x_2^{k_2} \cdots x_n^{k_n}$ exists if, and only if, the dimensional matrix of the variables $(x_1, x_2, \cdots,$

x_n) has the same rank as the dimensional matrix of the variables ($y, x_1, x_2, \ldots, x_n$).

The proof of this theorem follows immediately from algebraic principles. For, the condition that the product $y = x_1^{k_1}x_2^{k_2} \cdots x_n^{k_n}$ exist is tantamount to the condition that Equation 9 be consistent, since the exponents $k_1, k_2, \ldots, k_n$ (if they exist) are a solution of Equation 9. It is shown in algebra that Equation 9 is consistent if, and only if, the rank of the matrix of the coefficients of the k's is unchanged when the matrix is augmented by the column (a, b, c).

Theorem 6 will now be used to prove the following theorem.

Theorem 7. If $y = f(x_1, x_2, \ldots, x_n)$ is a dimensionally homogeneous equation, and if y is not dimensionless there exists a product of powers of the x's that has the same dimension as y.

In order to prove this theorem let us assume that $y = f(x_1, x_2, \ldots, x_n)$ is a dimensionally homogeneous equation, and that a product of powers of the x's with the dimension of y does not exist. It will be shown that this assumption leads to a contradiction. Therefore, the assumption must be inconsistent. This establishes the truth of the theorem.

The dimensional matrix is represented by Equation 5. Let the rank of this matrix be R. Since it is assumed that there is no product of the x's with the same dimension as y, Theorem 6 shows that the rank of the matrix that is obtained when the first column of the dimensional matrix is deleted is less than R. Hence, it may be assumed that a nonzero determinant of order R occurs in the left hand R columns of the dimensional matrix.

Let us first consider the case, $R = 3$. Then in view of the preceding remarks

$$\Delta = \begin{vmatrix} a & a_1 & a_2 \\ b & b_1 & b_2 \\ c & c_1 & c_2 \end{vmatrix} \neq 0$$

Let (α, β, γ) be the respective cofactors of the numbers (a, b, c) in this determinant. Then Laplace's development yields

$$\Delta = a\alpha + b\beta + c\gamma \neq 0 \qquad \text{(a)}$$

On the other hand,

$$a_i\alpha + b_i\beta + c_i\gamma = 0, \qquad i = 1, 2, \ldots, n \qquad \text{(b)}$$

This follows from the fact that Laplace's development of the following determinant yields

$$\begin{vmatrix} a_i & a_1 & a_2 \\ b_i & b_1 & b_2 \\ c_i & c_1 & c_2 \end{vmatrix} = a_i\alpha + b_i\beta + c_i\gamma$$

If $i = 1$ or $i = 2$, this determinant is automatically zero, since a determinant is zero if two of its columns are identical. If $i = 3, 4, \cdots, n$, the determinant is zero, since it has been shown that the rank of the dimensional matrix of the x's is less than three.

Since, by hypothesis, the function $y = f(x_1, x_2, \cdots, x_n)$ is dimensionally homogeneous, Equation 8 is an identity in the variables A, B, C. Accordingly, we may set

$$A = G^\alpha, \qquad B = G^\beta, \qquad C = G^\gamma$$

in which G is an arbitrary positive constant. Then Equation b yields, with Equation 7,

$$K_i = 1, \qquad i = 1, 2, \cdots, n$$

Hence, Equation 8 yields

$$Ky = f(x_1, x_2, \cdots, x_n) \tag{c}$$

where

$$K = G^{(a\alpha + b\beta + c\gamma)}$$

Now K is an arbitrary constant, since G is an arbitrary constant, and $a\alpha + b\beta + c\gamma \neq 0$. Equation c accordingly shows that there is no correspondence from the x's to y; i.e., $f(x_1, x_2, \cdots, x_n)$ is not a function. Since this is contrary to the hypothesis, the theorem is proved for the case in which the rank of the dimensional matrix is three.

If the rank R of the dimensional matrix is less than three, we consider only those rows of the dimensional matrix that contain a nonzero determinant of order R. The proof is then performed as in the preceding case, with the exception that the terms in the set (A, B, C) that correspond to the deleted rows of the dimensional matrix are now set equal to one.

From Theorem 7, it follows that a dimensionally homogeneous equation of the type $y = f(x_1, x_2, \cdots, x_n)$ may always be reduced to the form $\pi = F(x_1, x_2, \cdots, x_n)$, in which π is dimensionless. This form is obtained by dividing the equation by a product of the x's that has the same dimension as y. Such a product can usually be found quickly by inspection.

24. Reduction to Dimensionless Form

The observation that a dimensionally homogeneous equation among several variables can be reduced to an equation among a smaller number of dimensionless variables is principally due to Rayleigh and Buckingham. This article supplies a proof of this important principle.

It is necessary to call attention to the fact that the independent variables in a problem of dimensional analysis are always restricted to positive values. If this were not so, dimensionless products with fractional exponents would frequently be imaginary. However, there is a deeper reason for limiting the independent variables to positive values Only under this restriction is Buckingham's theorem logically valid

In the following discussion, the variables $(x_1, x_2, \ldots, x_n)$ will denote the independent variables in a problem of dimensional analysis These variables represent magnitudes of physical quantities such as velocity and momentum It will facilitate the discussion if geometrical terminology is employed Without attributing philosophical significance to multidimensional spaces, we may refer to the variables $(x_1, x_2, \ldots, x_n)$ as 'coordinates' in a space S It is an aid to the understanding of geometrical terminology if one visualizes what it means in two and three-dimensional spaces

Let A, B, C be any positive constants Let variables $K_1, K_2, \ldots, K_n$ be defined by Equation 7 The equations

$$x_i = K_i x'_i, \qquad i = 1, 2, \ldots, n \tag{a}$$

define a point transformation in the space S namely, a transformation that carries the point x'_i to the point x_i The point transformation that is defined by Equation a will be called a K transformation, and the set of all points that can be derived from a given point x'_i by K transformations will be called the K space that is generated by the point x'_i It may be readily seen that the resultant* of two K transformations is again a K transformation

Several preparatory theorems (called 'lemmas') which lead to Buckingham's theorem will now be proved In view of the preceding remarks, the variables x_i are restricted to positive values Hence the space S consists entirely of points whose coordinates are positive

Lemma 1 A K space is generated by any one of its points

Proof Consider the K space that is generated by the point x'_i Let x''_i be another point in this K space Then $x''_i = K'_i x'_i$ where

$$K'_i = (A')^{a_i}(B')^{b_i}(C')^{c_i}$$

If the point x_i also lies in the K space that is generated by the point x'_i,

$$x_i = K_i x'_i \qquad \text{where} \quad K_i = A^{a_i}B^{b_i}C^{c_i}$$

It follows that

$$x_i = \frac{K_i}{K'_i}x''_i = \bar{K}_i x''_i$$

where

$$\bar{K}_i = \left(\frac{A}{A'}\right)^{a}\left(\frac{B}{B'}\right)^{b}\left(\frac{C}{C'}\right)^{c_i}$$

Hence, the point x_i is derived from the point x''_i by a K transformation Thus, it is demonstrated that any point x_i that is derived from a point x'_i by a K transformation may also be derived from the point x''_i by a

*The set of all K transformations is a group in the algebraic sense

K-transformation, provided that the point x''_i lies in the same K-space as the point x'_i. This verifies the theorem.

Lemma 2. The space S is completely partitioned into nonoverlapping K-spaces.

Proof. This follows from the fact that any point in the space S generates a K-space and that a K-space is generated by any one of its points.

Lemma 3. A dimensionally homogeneous dimensionless function, $\pi = f(x_1, x_2, \cdots, x_n)$ is constant in any K-space.

Proof. Since π is dimensionless, its dimensional exponents a, b, c are all zero. Therefore, by Equation 7, $K = 1$. Also, since π is dimensionally homogeneous, it satisfies Equation 8. Hence,

$$\pi = f(K_1x_1, K_2x_2, \cdots, K_nx_n)$$

This equation means that the function π is constant in the K-space that is generated by the point $(x_1, x_2, \cdots, x_n)$.

From Lemma 3, it follows that any dimensionless product of the x's is constant throughout each K-space. Consequently, if $\pi_1, \pi_2, \cdots, \pi_p$ is a complete set of dimensionless products of the x's, there corresponds to each K-space of the space S a single set of values of the π's. The following lemma is the converse of this statement.

Lemma 4. If $\pi_1, \pi_2, \cdots, \pi_p$ is a complete set of dimensionless products of the x's, there corresponds to each set of values of the π's a single K-space of the space S.

Proof. Let $\pi'_1, \pi'_2, \cdots, \pi'_p$ be a set of constant values of the π's, and let x'_i and x''_i be two points of the space S that correspond to these values; i.e.,

$$\pi'_h = (x'_1)^{k_1^h}(x'_2)^{k_2^h} \cdots (x'_n)^{k_n^h} = (x''_1)^{k_1^h}(x''_2)^{k_2^h} \cdots (x''_n)^{k_n^h}$$

Since the x's have only positive values, we may take logarithms of both sides of this equation. Hence,

$$r_1k_1^h + r_2k_2^h + \cdots + r_nk_n^h = 0, \qquad h = 1, 2, \cdots, p \tag{b}$$

where

$$r_i = \log\left(\frac{x'_i}{x''_i}\right)$$

Now, since $\pi_1, \pi_2, \cdots, \pi_p$ is a complete set of dimensionless products, the exponents $k_i^1, k_i^2, \cdots, k_i^p$ are a fundamental system of solutions of Equation 10. This follows from Theorem 4. Since the solutions of Equation 10 are also solutions of Equation b, the coefficients in Equation b are linearly related to the coefficients of Equation 10; i.e., there exist constants α, β, γ, such that

$$\alpha a_i + \beta b_i + \gamma c_i = r_i = \log\left(\frac{x'_i}{x''_i}\right)$$

Letting the base of the logarithm be 10, we may write this equation in the form,

$$x'_i = x''_i(10)^{(\alpha a_i + \beta b_i + \gamma c_i)}$$

Let $A = 10^\alpha, \quad B = 10^\beta, \quad C = 10^\gamma$

Then the above equation becomes

$$x'_i = x''_i A^{a_i} B^{b_i} C^{c_i} = K x''_i$$

Thus, it is shown that the points x'_i and x''_i belong to the same K-space This verifies the theorem

The proof of Buckingham's theorem now follows immediately. By Theorem 7, any dimensionally homogeneous equation $y = f(x_1, x_2, \cdots, x_n)$ may be expressed in the form $\pi = F(x_1, x_2, \cdots, x_n)$, in which π is dimensionless By Lemma 4, there corresponds, to each set of values of $\pi_1, \pi_2, \cdots, \pi_p$, a single K-space By Lemma 3, there corresponds, to each K space, a single value of π Consequently, there corresponds, to each set of values of $\pi_1, \pi_2, \cdots, \pi_p$, a single value of π In other words, π is a single valued function of $\pi_1, \pi_2, \cdots, \pi_p$ Thus an arbitrary dimensionally homogeneous function $y = f(x_1, x_2, \cdots x_n)$ has been reduced to the form $\pi = F(\pi_1, \pi_2, \cdots, \pi_p)$ This is Buckingham's theorem By Theorem 5, $p = n - r$, in which r is the rank of the dimensional matrix of the x's

The converse of Buckingham's theorem is at once obvious, i e, an equation that relates dimensionless products is dimensionally homogeneous This observation is frequently regarded as adequate proof of Buckingham's theorem

Problems

1 Apply Theorem 1 to the equation $y = x_1x_2x_3$ What is the necessary and sufficient condition that is obtained for dimensional homogeneity of the equation?

2 Three dimensionless products are defined as follows

$$\pi_1 = x_1x_2^{1/2}x_4^{-1/2}x_5^{2}$$
$$\pi_2 = x_1^{-3/2}x_2^{2}x_3^{3/2}x_4^{-2}x_5^{4}$$
$$\pi_3 = x_1^{-4}x_2^{7/2}x_3^{6}x_4^{-13/2}x_5^{10}$$

By investigating the matrix of exponents ascertain if the three products are independent

3 The dimensions of five variables are given by the following matrix

	y	x_1	x_2	x_3	x_4
M	1	1	−1	2	0
L	3	−2	4	1	−1
T	2	−1	3	3	−1

Does a dimensionally homogeneous product of the following type exist?

$$y = x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4}$$

Explain

4. Are the following equations consistent? If so, determine the solution.

$$3x + y + 2z = 5$$
$$x + 2y + 4z = 2$$
$$x - y - 2z = 0$$

5. The dimensional matrix of three variables is given below. Determine a product of powers of the variables with the dimension $[M^{-3}L^2T^4]$. Does a product exist with the dimension $[MLT]$? Explain.

	x	y	z
M	2	2	5
L	1	−2	−3
T	2	5	0
Θ	0	−3	−2

6. The equation $y = f(x_1, x_2, x_3)$ is dimensionally homogeneous. The dimensional matrix of the variables is

	y	x_1	x_2	x_3
M	1	1	2	−1
L	3	−1	0	2
T	−2	−3	−2	2

In a model study of the relationship, x_1, x_2, and x_3 are reduced by the respective factors $\frac{1}{5}$, $\frac{1}{10}$, and $\frac{1}{4}$. By what factor is the dependent variable y reduced? (Hint. Use Theorem 1.)

CHAPTER 5

Similarity and Model Testing

> Experimenting with models seems to afford a ready means of investigating and determining beforehand the effects of any proposed estuary or harbor works a means, after what I have seen, I should feel it madness to neglect before entering upon any costly undertaking
>
> OSBORNE REYNOLDS

25 USE OF MODELS

Before an expensive engineering project is undertaken, it is sometimes advisable to study the performance of a small scale replica (model) of the system (prototype) that is to be built Model studies are performed in order to avoid costly mistakes and to obtain information that will aid in the design of the prototype Since it is relatively inexpensive to modify the construction of a model a "cut and try" method of design may sometimes be used, which would be excessively costly if it were undertaken with the full scale system

It must not be assumed, however, that model studies provide ready answers to all questions As a general rule, one cannot devise a suitable model test, nor can he interpret the results of a model test, unless he understands the basic theory of the phenomenon that he is studying Time and money are wasted by a test of a model that does not adequately represent the prototype The adage, "One test is worth a thousand expert opinions," is consequently a dangerous half truth Even though the general nature of a phenomenon is known, it is often impracticable to build a model that will furnish the desired information Furthermore, it is wasteful to resort to a model study, if the results can be predicted by theory, since the construction and testing of a model is usually expensive, compared to the cost of theoretical investigations and computations Not infrequently, the cost of a model study amounts to hundreds of thousands of dollars

In spite of their limitations, model tests have proved to be invaluable in many cases, and the use of models in engineering is steadily increasing It is impossible to survey the entire field of application of models, but the following examples will serve to indicate the extent of this field.

(*a*) *Hydraulic Structures.* The designs of nearly all major dams are checked, before construction, by tests of models whose sizes are usually 1/20 to 1/60 of the sizes of the prototypes. Not only are models of entire dams tested, but also models of various parts, such as stilling basins, spillways, penstocks, and gates, are studied, in order to obtain detailed information on the flow of water and its effects on the structure. Models are also used to study the performance of locks, flumes, conduits, and various other hydraulic structures.

(*b*) *Rivers and Harbors.* In the United States, and in other parts of the world, an immense amount of work is devoted to the dredging of rivers, straightening of channels, protection of banks and bottoms from erosion, construction of levees and floodways, and other forms of river control and improvement. Much of this work is planned on the basis of model studies of flood stages and scouring and shoaling characteristics. Estuaries and harbors present special problems of model design, since the currents and sediment transportation are here influenced strongly by sea tides. It is consequently necessary to duplicate the natural tidal cycles in the model. This calls for rather elaborate machinery. Wave action presents important harbor problems that can be investigated by means of models. These problems are concerned mainly with the effectiveness of proposed breakwaters for providing protection from waves, and with the damage that the waves may inflict on the breakwaters. Model studies of breakwaters are important, since breakwaters have occasionally failed to furnish the protection that was anticipated and have even had adverse effects in exceptional cases. The seriousness of such an engineering mistake is obvious when it is realized that a breakwater may cost several thousand dollars per running foot. The same remark applies to jetties, which are intended primarily to prevent shoaling of channels. Without a model study, it may be impossible to predict whether a proposed jetty will keep the channel open and whether it will cause undesirable shoaling or beach erosion in some other area.

Model tests of the type mentioned above are performed by the U. S. Waterways Experiment Station, the U. S. Department of Reclamation, and other agencies.

(*c*) *Hydraulic Machines.* Performance data for centrifugal pumps, hydraulic turbines, hydraulic torque converters, and other turbomachines may be approximately determined by tests of small-scale models. Models are consequently a valuable aid to designers of large turbomachines.

(*d*) *Airplanes.* Wind-tunnel testing plays an indispensable part in the design of any new airplane. Lift and drag coefficients of the wing, estimates of parasite drag, and other data are obtained in this way. Also, free-flight wind tunnels are used, in which the performances of flying models are observed. These tests show the response of the airplane to the controls, the facility of the airplane to pull out of a spin, and other characteristics Free flight tunnels are inclined, so that the weight of the model has an upstream component that provides the thrust Occasionally, radio-controlled motor driven flying models are tested in the natural atmosphere Take off, landing, and flight characteristics of flying boats have been studied in this way

Model airplanes are also used to study the interaction between aerodynamic forces and elastic deformations of the structures At high speeds, these interactions may produce destructive effects (wing flutter, torsional divergence, and aileron reversal) Wing flutter, the best known of these phenomena, is a violent forced vibration that occurs at a certain critical velocity

Model studies of airplanes are performed by the National Advisory Committee for Aeronautics, and by airplane manufacturers

(*e*) *Structures* Deflection tests and static destruction tests of structures or parts of structures are performed in order to predict how well the structures will fulfill their purposes These tests are sometimes performed on models if a prototype structure is not available or expendable, or if the testing of the prototype would be difficult or costly

(*f*) *Ships* Drag forces and wake patterns of naval vessels, flying boat hulls, and some commercial vessels are investigated by towing models by a power driven carriage that runs on a track above a canal in which the model floats Occasionally self propelled ship models are also used

26 Features of Models

If the parts of a model have the same shapes as the corresponding parts of the prototype, the two systems are said to be *geometrically similar* Geometric similarity is usually maintained in models of all types of fabricated structures However, for rivers, estuaries, and harbors, geometrically similar models, constructed to practicable scales, would ordinarily have water not more than one fourth of an inch deep Flow, under these conditions, would be strongly influenced by surface tension Also, movable bed models would not function satisfactorily, since the currents and eddies would not be strong enough to transport the sediment Consequently, models of rivers, harbors, and estuaries are frequently distorted, i e, the depths of water are relatively greater than in the prototypes

In a distorted model, the horizontal lengths and the vertical lengths are reduced by different scales Consequently, the planform is geometrically similar to that of the prototype, but the cross sections are distorted In geometrical terminology, the prototype of a distorted model is a "dilatation" of the model, or vice versa This signifies that the prototype is obtained by dilating the model to different scales in mutually orthogonal directions. The fact that the scale factor is usually constant for all horizontal directions is a special circumstance.

In general, there is a point-to-point correspondence between a model and its prototype. In geometrical terminology, two points that correspond to each other are *homologous*. The concept of homologous points leads immediately to the concept of homologous figures and homologous parts. Figures or parts of the model and the prototype are said to be "homologous" if they are comprised of homologous points.

If transient (i.e. time variable) phenomena occur in a model, it is necessary to introduce the concept of "homologous times." For example, the tidal period of a model of an estuary may be about 5 min, whereas the corresponding period for the prototype is about 12 hr. How, then, is the state of the model at a certain instant to be correlated with a state of the prototype? The answer, in this case, is readily perceived. In all cyclical phenomena, homologous times for a model and its prototype are instants that occur at the same fraction of a cycle.

The concept of similarity extends to many characteristics besides geometry. For example, it may be specified that the mass distribution in a model be similar to that in the prototype. This means that the ratio of masses of homologous parts shall be a constant that does not depend on the choice of the parts. In a restricted sense, this condition must be satisfied by an airplane wing-flutter model. The condition of similarity of mass distributions is not applied to all details of the structure, but it is required that the ratio of masses of segments of the wing and the model that are included between homologous cross sections shall be a constant. This condition is expressed by the statement that the spanwise distribution of mass of the model is similar to that of the prototype. Furthermore, the chordwise distributions of mass must be similar, to the extent that the centers of mass of homologous segments of the wing and the model are homologous points, and the mass moments of inertia of homologous segments of the wing and the model, with respect to the axis of twist, have a constant ratio. Another way of stating the latter condition is that the spanwise distributions of mass moments of inertia shall be similar.

It is important that the axes of twist in a wing-flutter model and in the prototype shall be homologous lines. Furthermore, the concept of similarity must be extended to stiffnesses; i.e., the ratio of stiffnesses of homologous cross sections of the wing and the model must be a constant. This constant should have the same value for torsional stiffness and bending stiffness, since the ratio of torsional stiffness to bending stiffness is an important factor for determining the flight speed at which flutter occurs.

27. Complete Similarity

A model study furnishes useful qualitative indications of the characteristics of the prototype. Usually, quantitative information is also sought. Models may be classified on the basis of the types of quantitative data that they are intended to supply. In many cases, the primary result of a model study is a single numerical value. For example, the main result of a destruction test of a structure is the ultimate load that the structure will sustain. The main result of a test of an airplane wing flutter model is the speed of flight at which flutter will occur.

The numerical value that is obtained by a test of a model depends on the values of the independent variables in the problem. A dimensional analysis of the relationship invariably leads to an equation of the form,

$$\pi = f(\pi_1, \pi_2, \ldots, \pi_p) \tag{a}$$

in which the π's are a complete set of dimensionless products. If we wish to know a particular value of π that corresponds to specified numerical values of $\pi_1, \pi_2, \ldots, \pi_p$ we may evidently achieve the result by means of a test of a model, provided that the independent dimensionless variables $\pi_1, \pi_2, \ldots, \pi_p$ have the same values for the model as for the prototype. The model and the prototype are then said to be *completely similar*. Since a complete set of dimensionless products determines all dimensionless products of the given variables, every dimensionless product has the same value for the model as for the prototype when complete similarity exists. Obviously complete similarity is impossible without geometric similarity.

Usually it is not feasible to impose complete similarity in a model test. Consequently some of the independent dimensionless variables, which are believed to have secondary influences or which affect the phenomenon in a known manner, are allowed to deviate from their correct values. An important part of the work of the model engineer—indeed the most important part—is to justify his departures from complete similarity or to apply theoretical corrections to compensate for them. For example, the influence of viscosity on the drag on a ship may be estimated by means of skin friction theory. Consequently, it is unnecessary to preserve the correct value of Reynolds' number in a towing test of a model (see Example 4).

One precaution, with regard to the neglecting of dimensionless products, must be mentioned. It may happen that forces that have practically no effect on the behavior of the prototype significantly affect the behavior of the model. For example, surface tension does not influence ocean waves, but, if the waves in a model harbor are less than one inch long, their nature is dominated by surface tension. Therefore, the Weber number is an important parameter for the model, although it is negligible for the prototype. Disturbing influences of this type are called *scale effects*. Surface roughness is sometimes a scale effect, since a surface that is practically smooth for the prototype may be relatively rough for the model. Scale effects occur, to some extent, in nearly all model tests. The best guard against them is to build models as large as is feasible.

Example 10. Model Ship Propeller

A ship with a propeller 20 ft in diameter is designed to move at a speed of 25 ft/sec when the propeller turns at a speed of 2 rev/sec. A 1/10 scale geometrically similar model of the hull and the propeller is to be tested in water, in order to determine the thrust force of the propeller. It is realized that complete similarity cannot be obtained, but viscosity is believed to have only a minor effect, and it is consequently decided to let Reynolds' number depart from its correct value. The problem is then to calculate the proper speed of rotation of the model propeller, the speed at which the model hull must move, and the percentage of reduction of Reynolds' number in the model.

The thrust force F of the propeller is determined by the rotational speed n (rev/sec) of the propeller, the diameter D of the propeller, the speed V of the ship, the acceleration of gravity g, the mass density ρ of the water, and the viscosity μ of the water. Gravity exerts an effect because the propeller creates surface waves. If the propeller is so deeply submerged that negligible waves are created, gravity does not affect the propulsion. Although the speed of the ship is determined by the rotational speed of the propeller, this is partly a characteristic of the hull. Consequently, we suppose that the speed of the model is controlled by other agencies than the propeller: e.g. by a towing carriage.

There are seven variables in the problem, and the rank of their dimensional matrix is three. Consequently, there are four dimensionless products in a complete set. A complete set of dimensionless products can be found by inspection. Let us choose the set that consists of the pressure coefficient $F/\rho V^2D^2$, the Froude number V^2/gD, the Reynolds number VD/ν, and the velocity ratio V/nD. Then, according to Buckingham's theorem, the thrust is given by an equation of the form,

$$F = \rho V^2 D^2 f\left(\frac{V^2}{gD}, \frac{VD}{\nu}, \frac{V}{nD}\right)$$

If viscosity is unimportant, complete similarity is insured by the condition that the products V^2/gD and V/nD have the same value for the model as for the prototype From the prototype data, the values of these products are

$$\frac{V^2}{gD} = \frac{(25)^2}{(32\ 2)(20)} = 0\ 9705, \qquad \frac{V}{nD} = \frac{25}{(2)(20)} = 0\ 625$$

Let primes refer to the model Then the condition for complete similarity is

$$\frac{(V')^2}{gD'} = 0\ 9705, \qquad \frac{V'}{n'D'} = 0\ 625$$

Since the model scale is 1/10, $D' = 2$ ft Consequently, the above equations yield $V' = 7\ 90$ ft/sec and $n' = 6\ 32$ rev/sec

The Reynolds number of the prototype is $VD/\nu = 500/\nu$, and the Reynolds number of the model is $V'D'/\nu = 15\ 81/\nu$ Since the model is to be tested in water, ν is the same for the model as for the prototype Consequently the ratio of the Reynolds number of the model to the Reynolds number of the prototype is $15\ 81/500 = 0\ 0316$, i e , the Reynolds number of the model is only about 3 percent of the Reynolds number of the prototype It is necessary to resort to propeller theory or to experience to estimate the effect of this large reduction of Reynolds' number In this example, the influence of viscosity may be regarded as a scale effect

28 Model Laws

Let us recall the equation for the drag on a body in a stream of incompressible fluid (Equation 1)

$$F = \rho V^2 L^2 f\left(\frac{VL\rho}{\mu}\right) \tag{a}$$

Suppose that a geometrically similar model of the body is to be tested in a wind tunnel or a water tunnel Let primes refer to the model In order that the unknown function f shall have the same value for the model as for the prototype, the Reynolds numbers of the two systems must be equal, i e ,

$$\frac{VL\rho}{\mu} = \frac{V'L'\rho'}{\mu'}$$

This equation may be written,

$$K_V K_L K_\rho = K_\mu \tag{b}$$

in which $K_V = V'/V$, $K_L = L'/L$, etc The K's are called *scale factors* Equations a and b yield

$$K_F = K_\rho K_V{}^2 K_L{}^2 = \frac{K_\mu{}^2}{K_\rho} \tag{c}$$

Equations b and c are said to express the *model law* for a body that is immersed in a stream of incompressible fluid. According to Equation c, the drag force on the model equals the drag force on the prototype, if the two bodies are tested in the same fluid.

As another example, consider the form drag of a model of a ship (Example 4). The form-drag coefficients C''_D of the model and the prototype are equal, if the Froude numbers are equal. This condition yields the equation,

$$K_V^2 = K_L \tag{d}$$

The equation for the form drag F (namely, $F = C''_D \rho V^2 L^2$) then yields

$$K_F = K_\rho K_V^2 K_L^2 = K_\rho K_L^3 \tag{e}$$

Model laws may be expressed by statements, rather than by equations. For example, Equation d means that the velocity of a model ship should vary as the square root of the linear dimensions of the model. Then, assuming that the model is tested in water, we perceive, by Equation e, that the form drag varies as the cube of the linear dimensions of the model. These principles are known as "Froude's law."

Equation b shows that, if a small-scale model is tested in the same fluid as the prototype, the preservation of Reynolds' number requires that the stream velocity for the model be greater than that for the prototype. On the other hand, according to Equation d, the preservation of Froude's number requires the opposite condition. Consequently, it is usually not feasible to preserve simultaneously the proper values of Reynolds' number and Froude's number in a model test.

Evidently, any relationship among dimensionless products can be expressed in the form of a model law.

29. General Concept of Similarity

Consider two systems, one of which is called the "prototype" and the other the "model." Let us select two homologous rectangular Cartesian space reference frames (x, y, z) and (x', y', z'), which, respectively, serve to designate points in the prototype and in the model. Suppose that the two systems are "geared together" in such a way that homologous points and homologous times are defined by the equations:

$$x' = K_x x, \quad y' = K_y y, \quad z' = K_z z, \quad t' = K_t t \tag{11}$$

The constants K_x, K_y, K_z are the *scale factors* for lengths in the x, y, and z directions. If the model is geometrically similar to the prototype, $K_x = K_y = K_z = K_L$. For the usual type of distorted model, two of the length scale factors are equal; i.e., $K_x = K_y \neq K_z$. Then the ratio K_z/K_x is called the *distortion factor*.

The constant K_t is called the *time scale factor*. In a cyclic phenomenon, the time scale factor is the ratio of the cyclic periods of the two systems. In steady flow processes, the time scale factor may be interpreted to be the ratio of time intervals in which two particles describe homologous parts of their trajectories.

It is important to bear in mind that, in transient phenomena, simultaneous states of the two systems are *not* considered. Rather, states that occur at *homologous* times are contemplated.

It is occasionally useful to conceive homologous directions in the spaces (x, y, z) and (x', y', z'). These are defined to be the directions of homologous straight lines. If the systems are geometrically similar, homologous directions are identical, i.e., their corresponding direction angles are equal.

The general concept of similarity may be defined in terms of two abstract scalar functions $f(x, y, z, t)$ and $f'(x', y', z', t')$ as follows:

> *The function f' is similar to the function f provided that the ratio f'/f is a constant, when the functions are evaluated for homologous points and homologous times. The constant ratio $f'/f = K_f$ is called the scale factor for the function f.*

As a concrete example, suppose that f and f' are the absolute temperatures at homologous points of a prototype and its model. Then, if f'/f is constant, the two systems are said to be "thermally similar." Observe that the values of f and f' are referred to homologous times, t and t', if the temperature distributions change with time.

30 Kinematic Similarity

The science of kinematics is the theory of space-time relationships. The expression "kinematic similarity" consequently signifies "similarity of motions." In order to define kinematic similarity it is necessary to establish a one-to-one correspondence between the material particles of two systems. Two particles that correspond to each other will be said to be "homologous." Kinematic similarity, or similarity of motions, is now naturally defined, as follows:

> *The motions of two systems are similar, if homologous particles lie at homologous points at homologous times.*

Homologous points and homologous times have been defined by Equation 11. It may be demonstrated that, in similar motions, the velocity vectors and the acceleration vectors at homologous points and homologous times have homologous directions. Consequently, the stream lines of similar fluid motions are homologous curves.

If kinematic similarity exists, corresponding components of velocity or acceleration are similar. In fact, the scale factors for these quantities are easily derived. In the model, a particle experiences a displacement from the point (x', y', z') to the point $(x' + dx', y' + dy', z' + dz')$ in the time interval dt'. Consequently, the velocity vector is

$$u' = \frac{dx'}{dt'}, \qquad v' = \frac{dy'}{dt'}, \qquad w' = \frac{dz'}{dt'}$$

Hence, by Equation 11, the relationship between the velocities of homologous particles of systems with similar motions is

$$u' = \left(\frac{K_x}{K_t}\right) u, \qquad v' = \left(\frac{K_y}{K_t}\right) v, \qquad w' = \left(\frac{K_z}{K_t}\right) w$$

Accordingly, the scale factors for u, v, and w are, respectively:

$$\frac{K_x}{K_t}, \qquad \frac{K_y}{K_t}, \qquad \frac{K_z}{K_t} \tag{12}$$

Consideration of the second derivatives likewise leads to the conclusion that the scale factors for the x, y, and z components of acceleration are, respectively:

$$\frac{K_x}{K_t^2}, \qquad \frac{K_y}{K_t^2}, \qquad \frac{K_z}{K_t^2} \tag{13}$$

If attention is directed to geometrically similar systems, then $K_x = K_y = K_z = K_L$ = length scale factor. Then, by Equation 12,

$$\frac{K_L}{K_t} = K_V = \text{velocity scale factor}$$

Equation 13 now permits the acceleration scale factor K_a to be expressed in terms of K_V and K_L as follows:

$$K_a = \frac{K_V^2}{K_L} \tag{14}$$

31. Dynamic Similarity

Two systems are said to be *dynamically similar* if homologous parts of the systems experience similar net forces. Consider two systems with similar mass distributions; i.e., $m' = K_m m$, in which m' and m are the masses of homologous parts and K_m is a constant. By Newton's law, the total force on a particle of the model with mass m' is

$$F'_x = m'a'_x, \qquad F'_y = m'a'_y, \qquad F'_z = m'a'_z$$

A corresponding equation, without primes, applies for the prototype. Hence, if kinematic similarity exists, Equation 13 yields

$$\frac{F'_x}{F_x} = \frac{K_m K_x}{K_t^2}, \quad \frac{F'_y}{F_y} = \frac{K_m K_y}{K_t^2}, \quad \frac{F'_z}{F_z} = \frac{K_m K_z}{K_t^2} \tag{15}$$

These are the scale factors for the total force components on homologous particles. It has thus been shown that dynamic similarity exists, if the systems are kinematically similar, and the mass distributions are similar. Integration shows that Equation 15 is valid for the forces on any homologous finite masses of the systems, as well as for infinitesimal particles.

If the systems are geometrically similar, Equation 15 yields

$$K_F = \frac{K_m K_L}{K_t^2} \tag{16}$$

As an example, suppose that a one tenth scale model of an engine performs three times as many cycles per second as the prototype. At homologous times, the moving parts of the two engines have the same relative positions. The time scale factor is 1/3 since the model performs a cycle, or any fraction of a cycle, in one third of the time of the prototype. Since the length scale factor is 1/10, Equation 12 yields $K_V = 3/10$, i.e., at homologous times any particle of the model has 3/10 of the velocity of the homologous particle of the prototype. Hence by Equation 14, the acceleration scale factor is 9/10, i.e., at homologous times any particle of the model has 9/10 of the acceleration of the homologous particle of the prototype. If the model and the prototype are made of the same material, $K_m = K_L^3 = 1/1000$. Then Equation 16 shows that the force scale factor is 9/10 000. For example, the net force on the piston (force due to pressure in the cylinder plus force of the wrist pin) is only 9/10 000 as great for the model as for the prototype. However, there is no determinate relationship between the pressures in the cylinders of the two engines, since the pressures are governed by the resisting torques that are applied to the shafts.

Similarly, in cases of dynamically similar fluid motions, the force scale factor refers to the net forces on homologous masses of the two fluids, but not to the pressure forces on homologous surface elements, since the latter forces can be changed arbitrarily by changing the hydrostatic pressures on the systems.

Example 11. Model Law for Underwater Explosions

When a high explosive is detonated under water, it is converted almost instantaneously into gas. The initial pressure p_0 of the gas depends on the chemical nature of the explosive. For TNT, p_0 is about two million pounds per square inch.[10]

The explosion causes a spherical shock wave to be propagated through the water. The pressure p of the shock wave, at any instant, depends on the radius R of the wave front, the initial gas pressure p_0, the mass m of the explosive, the mass density ρ of water, and the bulk modulus E of water. The term E is the ratio of the pressure to the volumetric strain, and it consequently has the dimension of pressure.

The relationship among the variables is indicated by the equation,

$$p = f(p_0, R, \rho, E, m)$$

Dimensional analysis now yields

$$p = p_0 f\left(\frac{p_0}{E}, \frac{m}{\rho R^3}\right)$$

This equation yields the following model law:

$$K_{p_0} = K_E = K_p, \qquad K_m = K_\rho K_R^3$$

If attention is restricted to explosions in water, $K_E = K_\rho = 1$. Then the preceding equations reduce to

$$K_p = K_{p_0} = 1, \qquad K_m = K_R^3$$

For example,[10] 1 lb of TNT causes a maximum shock pressure of 2200 lb/in.^{2} at a distance of 7.5 ft from the explosion. Consequently, 1000 lb of TNT causes a maximum shock pressure of 2200 lb/in.^{2} at a distance of 75 ft from the explosion. In the latter case, the thickness of the shock wave is ten times as great as for the 1-lb charge, since all lengths are altered by the factor K_R. This may be verified by making a dimensional analysis for the thickness of the shock wave.

Since the speed of a compression wave does not depend on the wave length, the time scale factor must equal the length scale factor; i.e., $K_t = K_R$ (cf. Equation 12). Consequently, the duration of the surge of pressure is ten times as great for the 1000-lb charge as for the 1-lb charge.

32. Conditions for Similar Flows of Incompressible Fluids

In the differential equations of dynamics of incompressible fluids, the pressure p and the specific weight ρg do not occur separately, but only in the expression $p + \rho g Z$, in which Z is the elevation above any fixed datum plane. It is convenient to denote the expression $p + \rho g Z$ by the symbol P. The quantity P is constant throughout a stationary fluid. Consequently, changes of P are attributable to inertia forces. We shall call the function

[10] R. H. Cole, *Underwater Explosions*, Princeton University Press, 1948.

P "hydraulic head," although, in hydraulics, this term usually means the ratio $P/\rho g$

There is, of course, the possibility that the pressure enter explicitly into the boundary conditions of a flow problem However, if the form of the boundary is known, the pressure on the boundary determines the boundary values of the quantity P Under these conditions, the problem can be solved in terms of the function P, without direct reference to the pressure Consequently, the hydraulic head P replaces the pressure p in dimensional analyses of problems of flow of liquids with fixed boundaries Since g does not then appear in the list of variables, Froude's number is not among the dimensionless products that are obtained

The situation is different, if the liquid has a free surface, as in natural watercourses flumes, spillways, weirs, etc In these cases, the form of the free boundary is generally unknown Consequently, the specification of the pressure at the free surface (usually atmospheric pressure) does not determine the hydraulic head P at the free surface, since the elevation Z is unknown Gravity then plays an individual role The Froude number and the Reynolds number accordingly both appear in dimensional investigations of flow with a free surface In some cases, the Weber number also appears

The weight of a gas is usually negligible compared to the other forces that act on it Consequently, terms that depend on gravity may be omitted, as an approximation Of course this approximation is not admissible in meteorological studies nor in problems dealing with convection currents that are set up by thermal differences

Since gravity does not enter explicitly into the equations that define the flow of a liquid with fixed boundaries the velocity components (u, v, w) at any point are determined by the viscosity μ and the mass density ρ of the fluid, a length L that specifies the size of the system and a characteristic velocity V (e g the average velocity in a conduit) Accordingly, dimensional analysis yields

$$u = Vf_1(\mathbf{R}), \qquad v = Vf_2(\mathbf{R}), \qquad w = Vf_3(\mathbf{R})$$

in which $\mathbf{R}$ is Reynolds' number The corresponding equations for a geometrically similar model are

$$u' = V f_1(\mathbf{R}), \qquad v' = V f_2(\mathbf{R}), \qquad w' = V' f_3(\mathbf{R})$$

Consequently, if the model and the prototype operate at the same value of Reynolds' number, and if the boundary conditions are kinematically similar

$$\frac{u'}{u} = \frac{v'}{v} = \frac{w'}{w} = \frac{V'}{V} = K_V$$

Accordingly, the flows are kinematically similar. Thus, the following principle is established:

> *A sufficient condition for kinematically similar flows of incompressible fluids with geometrically similar boundaries and kinematically similar boundary conditions is the equivalence of the Reynolds numbers.*

In flow of an incompressible fluid, dynamic similarity follows from kinematic similarity, since the mass distributions are necessarily similar (Article 31).

The above condition is also necessary for kinematic similarity, although, if violent turbulence occurs, Reynolds' number has only a small influence, since the shearing stresses due to viscosity are small compared to the shearing stresses due to momentum transport by turbulence. Since turbulent agitation increases with Reynolds' number, the influence of Reynolds' number always appears to approach an asymptote. Consequently, if Reynolds' number is sufficiently large, the effect of viscosity may be disregarded.

Turning now to the case of steady flow of a liquid with a free surface, we recall that gravity and surface tension may both affect the motion. Accordingly, dimensional analysis yields

$$u = Vf_1(\mathrm{R}, \mathrm{F}, \mathrm{W}), \qquad v = Vf_2(\mathrm{R}, \mathrm{F}, \mathrm{W}), \qquad w = Vf_3(\mathrm{R}, \mathrm{F}, \mathrm{W})$$

in which R, F, and W are Reynolds' number, Froude's number, and Weber's number. Consequently, in this case, kinematic similarity and dynamic similarity exist, if the boundary conditions are similar, and **R**, **F**, and **W** have the same values for the model as for the prototype. Although it is not always possible to realize these conditions, the effects of Reynolds' number and Weber's number are frequently small.

33. Conditions for Similar Flows of Gases

For definiteness, supersonic flow about an airfoil is considered. The reasoning is essentially the same for other types of flow. The velocity field is influenced, to some extent, by the temperature distribution at the boundary, and by the coefficient of thermal conductivity of the fluid. However, these effects are usually neglected. Then the velocity component u at any point is determined by the velocity V of the stream ahead of the airfoil, the pressure p_0 and the mass density ρ_0 of the undisturbed stream, a length L that specifies the size of the airfoil, and the kinematic viscosity ν. The speed of sound c_0 in the undisturbed region is given by the formula,

$$c_0 = \sqrt{\frac{1.4p_0}{\rho_0}}$$

Thus, p_0 may be calculated, if c_0 and ρ_0 are given Therefore, in the list of independent variables, p_0 may be replaced by c_0 A dimensional analysis of the problem accordingly yields

$$u = Vf_1(\mathrm{R}, \mathrm{M}), \qquad v = Vf_2(\mathrm{R}, \mathrm{M}), \qquad w = Vf_3(\mathrm{R}, \mathrm{M})$$

in which **R** and **M** are Reynolds' number and Mach's number It follows that the velocity field of the prototype is kinematically similar to the velocity field of a geometrically similar model if, and only if, the Reynolds number and the Mach number have the same values in either system

The mass density ρ at any point is given by an equation of the type,

$$\rho = \rho_0 f(\mathrm{R}, \mathrm{M})$$

Consequently, the mass distributions in the two systems are similar, if the conditions for kinematic similarity are satisfied Therefore, in compressible flows of gases, dynamic similarity is a consequence of kinematic similarity

It is rarely feasible to construct a model that preserves the correct values of both Reynolds' number and Mach's number However, if the velocity is small ($\mathrm{M} < \frac{1}{2}$), Mach s number has little effect on phenomena in a free stream of air (e g flow about an airfoil) On the other hand if the velocity is large ($\mathrm{M} > 0\ 80$), Mach s number has a pronounced effect Then the effect of Reynolds' number is usually small Strictly speaking, the thermal and dynamical properties of a real gas cannot be defined by a few characteristic constants and consequently other parameters than Reynolds' number and Mach's number may affect supersonic flow

34 Empirical Methods in Model Engineering

Geological phenomena, such as erosion and sediment transportation, are frequently studied by means of models These phenomena are so strongly influenced by mechanical properties of clays, rocks sediments, etc, that exact laws of similarity cannot be deduced Consequently, empirical or semiempirical interpretations of model data are employed

For example it might be desired to predict the progress in the scouring of the banks of a river during the next fifty years Suppose that actual detailed observations of the scour during the past twenty years are available The model engineer's approach to the problem is to construct a small scale model of a portion of the river, using some synthetic erodible material for the banks and the bottom Many different materials have been tried for this purpose The consistency of the material should be adjusted so that the scour that occurs in the model in the course of a few weeks duplicates the scour that has been observed in the prototype in the past twenty years This process is called ' verification" of the model It is reasonable to assume that, when verification has been achieved, the continued operation of the model for a few months will project the behavior of the prototype many years into the future.

Essentially the same procedure is used to predict the shoaling of tidal channels and harbors. Of course, an effort is made to produce flow in the model that is similar to that of the prototype, since similar bed movements can scarcely be expected, if the flows of water are dissimilar. Many questions of shoaling are raised by proposals for new structures, such as breakwaters and jetties. In these cases, the history of the prototype is known only for the period before the installation of the proposed structure. It is consequently necessary to bring the bed movement of the model into synchronism with the recorded bed movement of the prototype before the proposed structure is installed in the model. This method rests on the assumption that similarity of bed movements, once established, will not be disrupted by the installation of the new structure.

Empirical methods of model engineering provide the only known way of grappling with many difficult problems that are, at present, beyond the scope of rational analysis. The success of model studies of this type depends, to a large degree, on the skill and experience of the operator, and on the engineer's knowledge of the factors that influence the phenomenon that he is studying.

35. Use of the Principle of Similarity in Mathematical Investigations

Mathematical derivations are frequently encumbered by complicated functions of the constants in the problems. A preliminary dimensional analysis of a problem may reveal the ways in which some of the constants enter into the final solution. Then, if the problem is solved for the case in which these constants have specified numerical values (e.g. unity), the solution may be immediately generalized to cover the case in which the constants have arbitrary values.

For example, in calculating the behavior of an airfoil by conformal mapping, it is convenient to apply the mapping function to a circle that passes through the point -1 in the complex plane. This results in an airfoil whose chord is approximately 4 units. When the behavior of this airfoil is known, the characteristics of a geometrically similar airfoil with an arbitrary chord can be determined by the principle of similarity.

As another example, suppose that design charts for plate-stringer combinations of an airplane wing are constructed, for the case in which the stringers are 2 in. deep. From the knowledge of the relative strengths of geometrically similar structures, it is possible to use these charts for stringers of any depth.

Frequently, it is advantageous to seek the significant dimensionless products of a problem, before the mathematical analysis is carried out. Then the original differential equations may be expressed in terms of dimensionless notations. This method is particularly useful when numerical methods of solution are employed, since the result of a numerical analysis is greatly broadened by a reduction of the number of variables.

Problems

1. Two smooth balls of equal weight but different diameters are dropped from a balloon. The ratio of diameters is 3. What is the ratio of terminal velocities? (The terminal velocity is the velocity at which the air resistance balances the weight.) Are the flows completely similar? Explain. (Neglect compressibility.)

2. If, in Problem 1, the prototype is a ball 1 ft in diameter that weighs 5 lb and that falls in standard air ($\mu = 0.0001783$ poise), what is the weight of a solid steel ball (sp wt = 490 lb/ft³) that is dropped in water ($\mu = 0.01140$ poise) if the two cases are completely similar? If the terminal velocity for the prototype is 75 ft/sec, what is the terminal velocity of the steel ball? *Hint*: Note effect of buoyancy.

3. A $\frac{1}{10}$ scale flying model of a flying boat is completely similar to the prototype. The model takes off at a speed of 30 mi/hr. Neglecting viscosity, calculate the take-off speed of the prototype.

4. A model of a rubble breakwater is constructed of rocks that each weigh 2 lb. The rocks have the same specific gravity as those of the prototype. Appreciable damage to the model is observed if the wave height exceeds 1 ft. What is the minimum weight of each rock to be used in the prototype if the prototype is to withstand geometrically similar waves 20 ft high?

5. The outer surface of a model airplane wing is geometrically similar to the prototype. The scale factors for length, mass, and stiffness are $K_L = 1/10$, $K_m = 1/100$, $K_s = 1/400$. (The dimension of stiffness is (FL^2).) What is the ratio of natural frequencies of corresponding modes of vibration of the model and the prototype?

6. It is desired to test a half scale wind tunnel model at the same Mach number and the same Reynolds number as the prototype. If the prototype operates in standard air and the wind tunnel air is at standard temperature, what is the density of the wind tunnel air? (*Note*: The dynamic coefficient of viscosity and the speed of sound depend only on the temperature.)

7. It is desired to obtain the logarithmic decrement (i.e., the logarithm of the ratio of amplitudes of successive oscillations) of a ship that is rolling freely in a calm sea. The logarithmic decrement d depends on a length L that specifies the size of the ship, the moment of inertia I of the ship with respect to the axis of roll, the mass m of the ship, the kinematic viscosity ν of the water, the mass density ρ of the water, and the acceleration of gravity g. Derive the most general form of a dimensionally homogeneous equation for d. Why is it practically impossible to perform a small scale model test that preserves complete similarity?

8. For a model of an estuary, $K_x = K_y = 1/1000$, $K_z = 1/100$. The period from high tide to high tide is 6 min. What is the distortion factor? If kinematic similarity exists, what are the velocity scale factors K_u, K_v, K_w? What are the scale factors for the x, y, and z components of acceleration? For the x, y, and z components of force? What is the ratio of the speeds of flow at homologous points, if the direction cosines of the velocity vector in the model are $\frac{2}{3}$, $\frac{2}{3}$, $-\frac{1}{3}$? Assume that the tidal period of the ocean is 12 hr.

9. A concrete block (sp gr = 2.50) that weighs 200 lb in air slides on the bottom of a river when the current is 10 ft/sec. Assuming constant coefficient of friction, calculate the current required to move a geometrically similar block with sp gr 3.00 and weight 300 lb in air. Neglect viscosity.

10. A model of an airplane wing is to be tested in standard air, to determine the air speed V at which flutter occurs. The speed V depends on the mass density of the air, the mass of the wing, the stiffness of a cross section of the wing, and the length of the wing. The length scale factor is 1/10, and complete similarity is obtained. The air speed at which the model flutters is one half of the speed of flight at which the prototype flutters. What is the stiffness scale factor? (See Problem 5.)

11. A new form of flow meter, when tested in the laboratory, yielded a drop of pressure of 9 lb/in.^{2} for a flow of 3 ft^3/sec through a 6-in. pipe. If the same fluid is tested at the same temperature in a geometrically similar system, with a 24-in. pipe, what is the flow, if dynamic similarity is maintained? What is the pressure drop?

12. The gates of a canal lock extend the full height of the lock. When a vessel is being lowered in the lock, the gates at the outlet end are being opened at the rate of 10 in./min. The currents produced by the outflow of water cause the vessel to pull at its moorings. In a 1/25 scale geometrically similar model of the system, the maximum tension in the hawsers is 20 lb when the gates are opened at the proper rate. What is the correct rate of opening of the gates of the model? What is the maximum tension in the hawsers of the prototype? Neglect viscosity.

13. Wave motion in a horizontal canal is to be studied by means of a geometrically similar model. The model contains water, and the length scale factor is 1/5. Neglecting friction and surface tension, calculate the velocity scale factor, the time scale factor, the acceleration scale factor, and the force scale factor. Calculate the scale factor for the derivative of acceleration with respect to time. In the prototype, a wave travels a certain distance in 10 sec. In what time does the wave travel the homologous distance in the model?

14. A fluid is discharged from a nozzle in the form of a jet that breaks into a spray at a distance of 5 in. from the nozzle. The diameter of the jet is 0.03 in., and the velocity of the jet is 40 ft/sec. The surface tension of the fluid is 0.003 lb/ft. Another fluid with the same density and the same viscosity as the first fluid, but with surface tension equal to 0.005 lb/ft, is discharged from a geometrically similar nozzle. Calculate the scale factors for length, velocity, and time to insure kinematic similarity. What is the diameter of the second jet? The velocity of the second jet? At what distance from the nozzle does the second jet break into spray? What is the ratio of volumes of homologous droplets of spray? Which spray has the greater range? (Neglect gravity.)

15. A body with mass m executes damped oscillations under the action of a restoring force that is proportional to the displacement and a resisting force that is proportional to the square of the velocity. The ratio of the restoring force to the displacement is called the "spring constant," and the ratio of the resisting force to the square of the velocity is called the "damping constant." A small-scale model of the system is to be tested. The scale factors for mass, length, and time are respectively 1/100, 1/5, and 1/20. Calculate the scale factors for the spring constant and the damping constant.

16. The equation $y = f(x_1, x_2, x_3, x_4)$ is dimensionally homogeneous. The dimensional matrix of the x's is

	x_1	x_2	x_3	x_4
M	1	1	0	1
L	2	−2	−4	1
T	−2	−3	1	0

In a model study of the relationship, the variables x_1, x_2, x_3, and x_4 are reduced by the respective factors K_1 K_2, K_3 K_4 What relationship must the K's satisfy, if the reduction factor of y is such that complete similarity is obtained?

17 A $\frac{1}{16}$ scale model of a boat is constructed Neglecting viscosity, determine the correct scale factors for area, volume, weight, force, moments of force, moments of inertia, velocity, acceleration, angular velocity, rotational speed of the propeller, horsepower, and time The prototype weighs 100 000 lb, and it has 8000 hp Calculate the weight and the power of the model

CHAPTER 6

Dimensional Analysis Applied to Problems of Stress and Strain

36. Introduction

The difference between the number of original variables and the number of dimensionless products in a statical problem of stress analysis is usually two, since these problems involve only two dimensions: force and length. Although there are three dimensions if the mass system is used (by virtue of the relationship $[F] = [MLT^{-2}]$), the rank of the dimensional matrix, in any case, is not greater than two.

If two geometrically similar structures have similar loadings (in the sense that the loads on homologous parts have a constant ratio), they are said to have the same *type of loading*. If the type of loading on a structure is prescribed, all the loads are determined by a single force; e.g. the total load. However, in some instances, it is convenient to suppose that certain moments are applied to the structure, in addition to the forces. To be sure, a detailed specification of the forces on a structure determines the moments, but it may be useful to specify the moments in order to omit some of the details concerning the forces. For example, it is usually adequate to say that a certain moment is applied to the end of a beam, without specifying the exact distribution of force on the end. If moments and forces are considered to be distinct entities, the loads on a structure with a prescribed type of loading are determined by a single force F and a single moment M. Also, since a class of geometrically similar structures is contemplated, the size of a structure is determined by a single length L.

37. Large Deflections of Elastic Systems

In elasticity theory, deflections are understood to be "large," if the load-deflection relationship is essentially nonlinear. For example, the deflections of a flat plate are said to be large if membrane action is significant with respect to the bending action, since the membrane action introduces a nonlinear load-deflection relationship. Beam-column theory presents another example of large deflections. Likewise nonlinear relationships enter into ordinary elastic beam theory if the deflections are very large. In general the deflections of an elastic structure may be said to be "small" if and only if they are determined with sufficient accuracy by the classical linear theory of elasticity.

Insofar as dimensional analysis is concerned, large deflections require no special consideration. On the other hand an important simplification results if the deflections may be treated as small.

Consider an elastic system with a specified shape and a specified type of loading. Then according to the remarks of the preceding article, the size of the system and the load on each part are determined by a single length L, a single force F, and a single moment M. Hence any component σ of the stress at a specified point is determined by an equation of the type

$$\sigma = f(F, M, L, E, \nu)$$

in which E is Young's modulus and ν is Poisson's ratio. Likewise any component u of the deflection at a specified point is determined by an equation of the type

$$u = f(F, M, L, E, \nu)$$

Dimensional analysis of these relationships yields

$$\sigma = \frac{F}{L^2} f_1\left(\frac{F}{EL^2}, \frac{M}{FL}, \nu\right) \tag{a}$$

$$u = \frac{F}{EL} f_2\left(\frac{F}{EL^2}, \frac{M}{FL}, \nu\right) \tag{b}$$

These equations yield the following general model law for statically loaded elastic structures:

$$\begin{aligned} K_\nu &= 1 \qquad K_F = K_E K_L^2 \qquad K_M = K_F K_L \\ K_\sigma &= \frac{K_F}{K_L^2} = K_E \qquad K_u = K_L \end{aligned} \tag{17}$$

in which the K's are scale factors (Article 28). Since Poisson's ratio does not enter into analyses of trusses and other frames, it is unnecessary to maintain the condition $K_\nu = 1$ in model studies of this type of structure.

Example 12. Model of an Archery Bow

In order to design an aluminum alloy archery bow for commercial purposes, a 3/4 scale hickory model was made. A force of four pounds was required to draw the model bow the length of a model arrow. The moduli of elasticity of hickory and aluminum alloy are respectively 2.1×10^6

lb/in.² and 10.5×10^6 lb/in.². It is required to calculate the force necessary to draw the prototype bow the length of an arrow.

The given data are $K_L = 3/4$ and $K_E = 2.1/10.5 = 1/5$. The condition, $K_u = K_L$, is automatically satisfied, since the length scale factor for the arrows is the same as for the bows. Equation 17 accordingly yields

$$K_F = \left(\frac{1}{5}\right)\left(\frac{3}{4}\right)^2 = \frac{9}{80} = \frac{F'}{F}$$

in which the prime refers to the model. Since $F' = 4$ lb, it follows, $F = 35.5$ lb; i.e., the force required to draw the aluminum-alloy bow is 35.5 lb.

38. Statical Loading Beyond the Yield Point

If a material is loaded above the yield point, permanent set results when the load is relieved. Since, within certain limits, any specified amount of set can be realized, there is no unique relation between stress and strain in the inelastic range. However, the strains are determined by the stresses, if the stresses increase slowly and monotonically* during the loading, and appreciable creep does not occur. Attention will be restricted to this case.

According to the Hencky-von Mises theory of plasticity, the general stress-strain relationship of a material, under monotonic increasing loading, is determined by the ordinary tension stress-strain curve, and by the value of Poisson's ratio in the elastic range. To plot the tension stress-strain curve of a material in a dimensionless form, we may let the ordinate be σ/E, in which σ is the true tensile stress and E is the value of Young's modulus in the elastic range. Two materials will be said to have the same *type of stress-strain relationship*, if their dimensionless stress-strain curves are identical. If the type of stress-strain curve of a material is preassigned, the stress-strain relationship is determined completely by the elastic modulus E and Poisson's ratio ν. Analogously, the geometry of a system with a given shape is defined by a single length L. Accordingly, when attention is restricted to materials with the same type of stress-strain relationship (in particular, when all structures are made of the same material), the distinction between elastic materials and inelastic materials is eliminated, insofar as dimensional analysis is concerned. Hence, the model law (Equation 17) remains valid for the strain-hardening range. This law may be expressed in words, as follows:

If the linear dimensions of a structure are changed by a factor k, the applied forces are changed by the factor k^2, and the applied moments are

* In mathematics, a function is said to increase monotonically if a positive increment of the independent variable never causes a decrease in the value of the function.

changed by the factor k^2, the deflections are changed by the factor k and the stresses are unchanged

This is a remarkably general law, for it is independent of the stress strain curve of the material, and independent of the magnitudes of the deflections It should be observed, however, that, when the weight of the structure contributes appreciably to the load, the conditions of the theorem cannot be satisfied, since an alteration of the linear dimensions by the factor k necessarily alters the weight by the factor k^3

Ordinarily, the stress that causes failure of a structure (fracture, yielding, buckling, etc) does not depend appreciably on the size of the structure For example, it has been found experimentally that statically loaded geometrically similar joints, made of 24S T aluminum alloy sheets and 17S T rivets, fail at practically the same stress, irrespective of their size [11] On the other hand, some observers have noted appreciable scale effects in statically loaded notched tensile specimens of magnesium

Endurance limits of materials exhibit pronounced scale effects It is found that the theoretical stress concentration factors of elasticity theory are much too large to account properly for the endurance limits of small notched fatigue specimens However, if the specimens are so large that many millions of crystallites lie in the regions of high stress concentration, the endurance limits approach those of unnotched specimens, if the theoretical stress concentration factors are employed The cause of scale effects in fatigue is not clearly understood However, since the crystallites are rarely scaled in the same proportion as the macroscopic dimensions of specimens, the existence of scale effects is not surprising

In impact tests, scale effects sometimes result from the fact that all specimens are not loaded at the same rate, since fracture stresses may be influenced by the rate of stressing

Example 13 Wind Loads on Large Windows

Recent architectural proposals for unusually large windows have raised the question of allowable wind pressures on windows Since the deflection of a large window may be many times its thickness, the question belongs to the class of large-deflection problems of plates

A large window is supported by fixtures that offer no restraint to rotation of edges of the window Although there is some resistance to linear deflections of the edges, the beams that support the edges may have appreciable flexibility

[11] R L Fefferman and H L Langhaar, Investigations of 24S T Riveted Tension Joints, *J Aeronaut Sci* Vol 14, no 3 Mar 1947

To make a model study of the problem, a small horizontal pane of glass, supported in the same manner as the prototype, may be loaded with sand until it breaks. Let us consider the model law for this type of test.

A class of geometrically similar windows is considered. It is important to recognize that the same-scale factor applies for the thickness as for the width and the height. This requirement sometimes introduces a practical difficulty in tests of this type, since sheets that are thin enough to insure geometrical similarity may be unavailable.

Assuming that geometrical similarity of the panes of glass is obtained, we may specify the size of the window by a single length L. Any stress component σ at any specified point of the glass is determined by the wind pressure p, the characteristic length L, the modulus of elasticity E of glass, Poisson's ratio ν, and the flexural stiffness $B = E_b I_b$ of the beams that support the edges. Dimensional analysis accordingly yields

$$\sigma = pf\left(\frac{p}{E}, \frac{B}{EL^4}, \nu\right)$$

The resulting model law is

$$K_\nu = 1, \qquad K_p = K_E, \qquad K_B = K_E K_L^4, \qquad K_\sigma = K_p$$

Since the model is to be made of glass, $K_\nu = K_E = 1$. Consequently, the equations reduce to

$$K_p = K_\sigma = 1, \qquad K_B = K_L^4$$

Accordingly, the stiffness of the supporting beams should vary as the fourth power of the linear dimensions of the glass. When this condition is satisfied, the model and the prototype experience the same stresses, if the applied pressures are the same. If the weight of the glass model is significant, it may be regarded as a part of the applied load. Since the model is made of glass, it will break at the same stress as the prototype—barring differences in the physical properties that result from manufacturing processes.

39. Bending of Ductile Beams*

Consider a class of ductile prismatic or cylindrical beams with similar cross sections that are symmetrical with respect to two perpendicular axes. Let a beam be bent in a plane of symmetry by a bending moment M, and let the size of the cross section of the beam be specified by the distance c from the neutral axis to the outermost fiber. Consider beams of the same material, and assume that the stress-strain curves for tension and compression

*This material is adapted from a master's thesis by W. B. Sanders, Jr., which was submitted to the Department of Theoretical and Applied Mechanics of the University of Illinois.

are identical Let the curvature of the neutral surface, due to bending, be $1/r$ Then,

$$1/r = f(M, c, E)$$

Dimensional analysis now yields

$$\frac{c}{r} = f\left(\frac{M}{Ec^3}\right) \tag{a}$$

The theory of elastic beams suggests that, instead of the variable M/Ec^3, we should preferably employ the variable Mc/EI, in which I is the moment of inertia of the cross section about the neutral axis This change of variable is permissible, since I/c is proportional to c^3, if the shape of the cross section is given Accordingly, Equation a may be written

$$\frac{c}{r\epsilon_y} = f\left(\frac{Mc}{EI\epsilon_y}\right) \tag{b}$$

in which ϵ_y is the strain at the yield point of the material The introduction of the factor ϵ_y results in convenient numerical values of the variables. A more significant reason for introducing this factor will appear later

Equation b shows that a curve may be plotted with abscissa $Mc/EI\epsilon_y$ and ordinate $c/r\epsilon_y$ In the elastic range $c/r\epsilon_y = Mc/EI\epsilon_y$, and therefore the first part of the curve is a 45° straight line Furthermore, the curve is asymptotic to a vertical line, since there is an ultimate bending moment for a ductile beam The general form of the curve is shown in Figure 5

In order to plot the curve accurately we make use of the equation of statics,

$$M = 2\int_0^c b\sigma z\, dz \tag{c}$$

in which z is the ordinate erected from the neutral axis of the cross section, b is the width of the cross section at ordinate z, and σ is the stress at ordinate z Set $z = c\zeta$ and $b = c\beta$ Then Equation c becomes

$$M = 2c^3\int_0^1 \beta\sigma\zeta\, d\zeta$$

The strain at ordinate z is $\epsilon = z/r = c\zeta/r$ Set

$$\frac{\sigma}{E} = f\left(\frac{\epsilon}{\epsilon_y}\right) = f\left(\frac{c\zeta}{r\epsilon_y}\right)$$

Then

$$\frac{Mc}{EI\epsilon_y} = \frac{2c^4}{I\epsilon_y}\int_0^1 \beta\zeta f\left(\frac{c\zeta}{r\epsilon_y}\right) d\zeta \tag{d}$$

The function $f(c\zeta/r\epsilon_y)$ is represented by a dimensionless stress-strain curve with ordinate σ/E and abscissa ϵ/ϵ_y (Figure 3). Also, the function $\beta(\zeta)$ is determined by the shape of the cross section of the beam. Consequently, the integral in Equation d may be regarded as a known function of $c/r\epsilon_y$. For a given value of $c/r\epsilon_y$, this function may be determined by evaluating the integral numerically. Accordingly, since c^4/I is determined by the shape of the cross section, the value of $Mc/EI\epsilon_y$ corresponding to any value of $c/r\epsilon_y$ may be calculated by Equation d. Hence, the curve of $c/r\epsilon_y$ versus $Mc/EI\epsilon_y$ may be plotted.

The curve does not depend on the width of the cross section. For, if the width b is changed by a constant factor, the terms β and I are also changed by this factor, and therefore the right side of Equation d remains unchanged. For example, the curve for circular cross sections is also valid for all elliptical

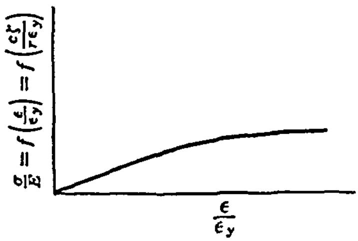

FIG. 3. Dimensionless Stress-Strain Curve

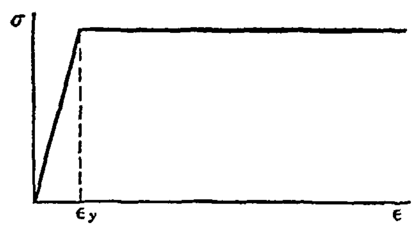

FIG. 4. Simplified Stress-Strain Curve

cross sections, regardless of their sizes and their eccentricities, since any ellipse may be obtained by widening or narrowing a circle by a constant factor. Likewise, a single curve suffices for all rectangular cross sections, regardless of their sizes and their aspect ratios. A discontinuity may appear in the curve, if the compression side of the beam buckles.

Frequently, the simple type of stress-strain curve shown in Figure 4 is used in analyses of plastic deformation. Then,

$$\frac{\sigma}{E} = f\left(\frac{c\zeta}{r\epsilon_y}\right) = \begin{cases} \left(\dfrac{c\zeta}{r\epsilon_y}\right)\epsilon_y, & \zeta < \zeta_1 \\ \epsilon_y, & \zeta > \zeta_1 \end{cases}$$

in which $\zeta_1 = r\epsilon_y/c$. Accordingly, Equation d yields

$$\frac{Mc}{EI\epsilon_y} = \frac{2c^4}{I}\left[\frac{1}{\zeta_1}\int_0^{\zeta_1} \beta\zeta^2\, d\zeta + \int_{\zeta_1}^1 \beta\zeta\, d\zeta\right] \tag{e}$$

In view of Equation e, the curve of $c/r\epsilon_y$ versus $Mc/EI\epsilon_y$ does not depend on the yield point of the material, provided that the stress-strain curve is of the type shown in Figure 4 Likewise, the curve is independent of the width and the size of the cross section

For a rectangular cross section, β is a constant Consequently, the integrals in Equation e are easily evaluated The following equation is then obtained

$$\frac{Mc}{EI\epsilon_y} = \begin{cases} \dfrac{3}{2} - \dfrac{1}{2}\left(\dfrac{r\epsilon_y}{c}\right)^2, & \dfrac{c}{r\epsilon_y} > 1 \\ \dfrac{c}{r\epsilon_y}, & \dfrac{c}{r\epsilon_y} < 1 \end{cases} \tag{f}$$

This equation is represented by Figure 5 This curve is valid for all ductile beams of rectangular cross section, regardless of their sizes, their aspect ratios and their yield stresses, provided that the stress-strain curve is of the type shown in Figure 4 A similar curve may be plotted for all ductile beams of elliptical cross section

If the deflections of a beam are not too large the curvature $1/r$ is closely approximated by d^2y/dx^2, in which x is the axial coordinate and y is the *deflection* Hence,

$$\frac{c}{r\epsilon_y} = \frac{c}{\epsilon_y}\frac{d^2y}{dx^2} = f\left(\frac{Mc}{EI\epsilon_y}\right) \tag{g}$$

Let L be the length of the beam or some specified fraction of the length Introduce the dimensionless notations

$$x = \xi L, \qquad y = \frac{\eta L^2 \epsilon_y}{c}$$

Then Equation g becomes

$$\frac{d^2\eta}{d\xi^2} = f\left(\frac{Mc}{EI\epsilon_y}\right) \tag{h}$$

For statically determinate beams $Mc/EI\epsilon_y$ is a known function of ξ Dimensionless constants of the type PcL/EI appear in this function, in which P denotes a load on the beam It is advisable to denote these constants by symbols, such as k_1 and k_2 For specified values of these constants, Equation h may be integrated numerically with the aid of Figure 5, which defines the function $f(Mc/EI\epsilon_y)$ Thus the deflection curve of a statically determinate ductile beam is determined

Problems of redundant beams are closely related to the problem of deflections If a beam has several immovable supports the reactions are determined by the condition that the deflections are zero at the supports However, detailed solutions of problems of ductile redundant beams are very tedious In some cases the redundancies are eliminated if the ultimate bending moment is known to exist at certain sections. This matter has been discussed by Van den Broek.[12]

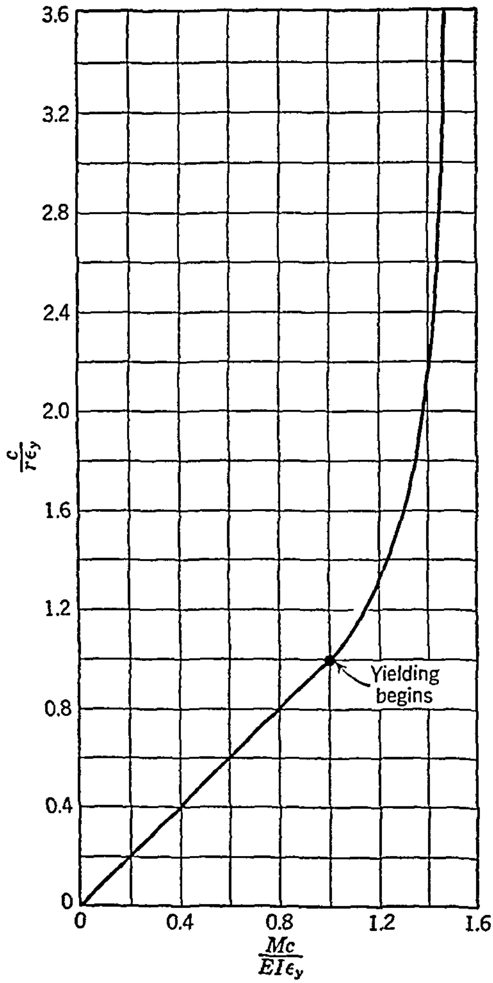

FIG. 5. Dimensionless Graph Showing Relationship between Bending Moment and Curvature for All Ductile Rectangular Beams with the Type of Stress-Strain Diagram Shown in Figure 4

40. DIMENSIONLESS PLOTTING OF TEST DATA FOR RIVETED JOINTS*

The bearing stress s that a rivet exerts on a sheet is defined by the equation, $s = F/td$, in which F is the load on the rivet, d is the diameter of the rivet, and t is the thickness of the sheet. If the bearing stress becomes too

[12] J. A. Van den Broek, *Theory of Limit Design*, John Wiley, New York, 1948.
*This article is adapted from the material in Reference 11.

great, the rivet shears or else the rivet hole is elongated by crushing of the sheet behind the rivet The latter phenomenon is known as a "crushing failure" If rivets lie near a free edge, a type of failure known as "tear out" may occur This is not clearly distinguishable from crushing It may be included in the category of crushing failures, if the distance e from the line of rivets to the edge of the sheet is regarded as one of the variables that influences crushing The bearing stress that causes crushing failure or shear failure depends also on the thickness t of the sheet, the diameter d of the rivet, and on properties of the materials Although various proper ties of the materials may be influential, all of the characteristic stresses bear

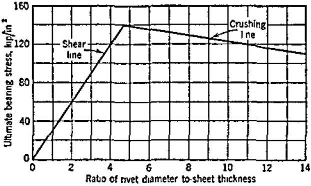

FIG 6 Experimental Chart Showing Ultimate Bearing Stress for 17S-T Protruding Head Rivets in Unclad 24S-T Sheets

Crushing line applies for joints with edge distance equal to two rivet diameters By courtesy of the Institute of Aeronautical Sciences (Reference 11)

a constant ratio to any one of them Consequently, the material properties are designated by a single characteristic stress—say the ultimate tensile stress σ_u of the sheet material Accordingly, the ultimate bearing stress s_u for the rivets is given by an equation of the form

$$s_u = f(\sigma_u, t, d\ e)$$

Dimensional analysis yields

$$s_u = \sigma_u f\left(\frac{d}{t}, \frac{e}{d}\right) \tag{a}$$

G Holback[13] is the first investigator who recognized the importance of this type of relationship He plotted dimensionless data for 24S T and A17S T aluminum alloy rivets at three diameters edge distance Feffer-

[13] G Holback The Structural Analysis and Significance of Rivet Shear Tests, *Proc Soc Exp Stress Anal* Vol III no 1 1945

man[11] plotted similar data for 17S-T rivets in 24S-T sheet at two diameters edge distance and obtained the graph shown in Figure 6. Note that the graph consists of two lines: one line for shear failures of the rivets and one for crushing failures of the sheet. Holback observed that, in some instances, these two lines are joined by a short path that cuts off the sharp corner. For a specified type of rivet (flat-head, countersunk, etc.), and for given materials, a graph, such as Figure 6, furnishes complete information concerning the ultimate shearing stresses and the ultimate bearing stresses of rivets.

Let us now turn attention to riveted joints. From the standpoint of stress analysis, lap joints and butt joints are equivalent, since a butt joint is merely two lap joints in tandem—the laps being formed with a common splice plate. Only single lap joints will be considered in the following. It may be assumed that one of the plates is somewhat thinner than the other. The thicker plate may be regarded as a part of the test jig.

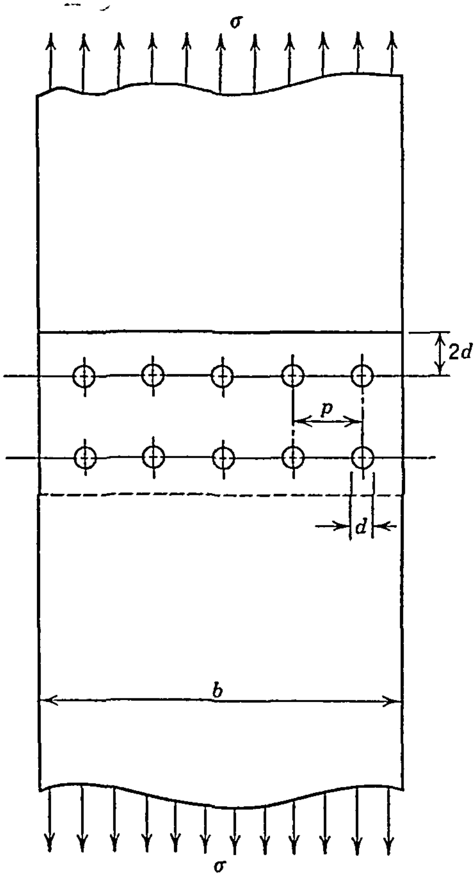

FIG. 7. Two-Row Tandem-Riveted Lap Joint

Although the rivet pattern is of no significance in the present discussion, let us consider, for definiteness, the rivet pattern shown in Figure 7. Suppose that the rivet spacing p, the rivet diameter d, and the sheet thickness t may be varied independently. Let σ_j be the value of the sheet stress σ that causes failure of the joint. Then, letting σ_u (the ultimate tensile stresss of the sheet material) be the characteristic stress to which other material properties are referred, we have

$$\sigma_j = f(\sigma_u, p, d, t)$$

Dimensional analysis yields

$$\sigma_j = \sigma_u f\left(\frac{d}{p}, \frac{d}{t}\right) \tag{b}$$

Instead of d/p, we may introduce the so called "rivet factor,"

$$C = \frac{b - nd}{b}$$

in which b is the width of the joint and n is the number of rivets on the front row. This is the ratio of the cross sectional area of the sheet on the front line of rivets to the cross-sectional area of the sheet at a section that does

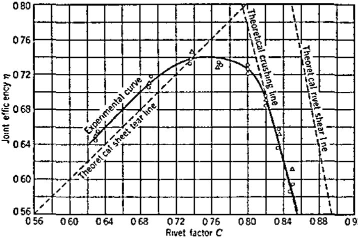

Fig 8 Efficiencies of Two-Row Tandem Riveted 24S-T Lap Joints with Protruding Head 17S T Rivets $d/t = 6\ 25$

By courtesy of the Institute of Aeronautical Sciences (Reference 11)

not contain rivet holes Also, it is convenient to introduce the joint efficiency η, defined by $\eta = \sigma_j/\sigma_u$ Then, Equation b becomes

$$\eta = f\left(C, \frac{d}{t}\right) \tag{c}$$

Figure 8 is a graph of Equation c for two-row tandem 24S T aluminum alloy joints with 17S-T flat head rivets that all have the same diameter (Figure 7) The theoretical graph for ideal joints consists of the two dotted straight lines The 45° line, $\eta = C$, corresponds to tearing of the sheet on the front line of rivets The rivet shear line and the crushing line both belong to the family of straight lines through the point $C = 1$, $\eta = 0$ These two lines are determined by the elementary theory of riveted joints [11] The crushing line depends only slightly on ratio d/t, but the rivet shear line is strongly dependent on this ratio (see Figure 6) Since, in Figure 8, the shear line lies to the right of the crushing line crushing precedes shear, and therefore shear is impossible. This is as it should be, for shearing of the rivets is an indication of inefficient design. A properly designed joint usually fails by a tear of the sheet on the front line of rivets. The theoretical lines of sheet tear, rivet shear, and crushing are useful guides for plotting experimental data. They form a framework that indicates approximately where experimental points should fall.

A graph such as Figure 8 may be plotted for joints with any given rivet pattern, since Equation c is valid, irrespective of the rivet pattern. Graphs for aluminum-alloy joints with various rivet patterns are given in Reference 11.

41. Small Deflections of Elastic Structures

The stresses in the members of a statically determinate structure obviously do not depend on the elastic modulus. This statement remains true for statically indeterminate structures, provided that all members have the same elastic modulus, and provided that second-order effects, such as beam-column action, are insignificant.

Since the elastic modulus does not affect the stresses in a slightly deflected elastic structure, the stress σ at any specified point is determined by a single length L, a single force F, and a single moment M, it being assumed that the shape of the structure and the type of loading are preassigned. The moment M may be disregarded, if we consider that all moments are determined by the applied forces. Then the only dimensionally homogeneous equation among the variables is

$$\sigma = \frac{kF}{L^2} \tag{a}$$

in which k is a dimensionless constant that possibly depends on Poisson's ratio. This equation shows that the stress at any point of a slightly deflected elastic structure is proportional to the total load on the structure.

Now the deflections of an elastic structure are proportional to the strains, and the strains, in turn, are proportional to the stresses. Consequently, the preceding conclusion yields the following principle:

Slight deflections of an elastic structure are proportional to the total load on the structure.

In other words, the first part of the load-deflection graph for an elastic structure is a straight line.

In view of this principle, Equation b of Article 37 must take the more special form,

$$u = \frac{kF}{EL} \tag{b}$$

in which k again is a dimensionless constant that may depend on Poisson's ratio

Equations a and b furnish the following model law for elastic structures with small deflections

$$K_\nu = 1, \qquad K_\sigma = \frac{K_F}{K_L^2}, \qquad K_u = \frac{K_F}{K_E K_L} \tag{18}$$

Note that these equations are consistent with the corresponding equations for large deflections (Equation 17), but that they are less restrictive, since the condition $K_F = K_E K_L^2$ is eliminated According to Equation 18, the load that is placed on a model of a structure is arbitrary

Example 14 Stresses in an Arched Dam

A 1/50 scale model of an arched dam is made of a cast resin that has about the same Poisson ratio as concrete A partition is placed across the model canyon just upstream from the dam, and the space between the partition and the dam is filled with mercury in order that hydrostatic pressure distribution is automatically obtained Strains in the model are determined by electric strain gages

Since the depth of mercury at any point of the model is only 1/50 of the depth of water at the homologous point of the prototype, and since the specific gravity of mercury is 13.6, the pressure scale factor is $K_p = 13.6/50 = 0.272$ Since the area scale factor is $K_L^2 = 1/2500$, the force scale factor is

$$K_F = K_p K_L^2 = \frac{0.272}{2500} = 0.000109$$

Hence, by Equation 18, $\qquad K_\sigma = \frac{K_F}{K_L^2} = 0.272$

Accordingly, the stresses in the model are 27.2 percent of the homologous stresses in the prototype

Since the stress analysis of an arched dam consumes many thousands of man hours, and since the computations employ approximations of questionable accuracy, model studies have potential value in this field

Example 15 Stresses in an Airport Pavement

In the analysis of rigid pavements, it is commonly assumed that the subgrade (i.e. the material beneath the concrete slab) experiences a vertical deflection that is proportional to the pressure that the slab imparts to it The ratio k of the pressure to the deflection of the subgrade is presumed to be a characteristic constant, called the "subgrade modulus." Evidently, the dimension of the subgrade modulus is $[FL^{-3}]$.

If a wheel rests on a concrete slab, the lower side of the slab is placed in tension, as a result of the bending. If the tensile stresses are too great, the slab cracks. It is consequently important to calculate the maximum tensile stress σ that results from a given wheel load. The primary variables that determine σ are the subgrade modulus k, the elastic modulus E of the concrete, the thickness h of the concrete, the wheel load F, and the air pressure p in the tire. Hence,

$$\sigma = f(F, p, h, k, E)$$

Dimensional analysis yields

$$\sigma = \frac{F}{h^2} f(\pi_1, \pi_2, \pi_3)$$

in which $\pi_1 = \frac{ph^2}{F}, \qquad \pi_2 = \frac{E}{kh}, \qquad \pi_3 = \frac{p}{E}$

If the load F is increased, the area of contact between the tire and the pavement is increased. Accordingly, the type of loading changes with the magnitude of the load. Under this condition, the stresses are not directly proportional to the load. Rather, the spreading of the load causes the rate of increase of σ with respect to F to decline as F increases. Consequently, f is an increasing function of π_1. Also, f is an increasing function of π_2, for, if the subgrade modulus k is increased, the subgrade gives greater support to the pavement. Likewise, f is an increasing function of p, for, as the air pressure in the tire is increased, the load becomes more concentrated.

These conditions are satisfied, if f is a function of the single variable $\pi_1^2\pi_2$; i.e.,

$$\sigma = \frac{F}{h^2} f\left(\frac{Ep^2h^3}{kF^2}\right) \tag{a}$$

This simple form was derived by Westergaard[14] by means of principles of elasticity theory. In view of Equation a, the results of Westergaard's theory can be presented by a single curve[15] with abscissa Ep^2h^3/kF^2 and ordinate $\sigma h^2/F$. Apparently, this fact has not been generally recognized, for lengthy tabulations of Westergaard's results, showing the separate effects of the

[14] H. M. Westergaard, Stresses in Concrete Pavements Computed by Theoretical Analysis, *Public Roads*, Vol. 7, no. 2, p. 25, Apr. 1926.

[15] This is also true of a revised theory that Westergaard developed to take account of inelastic action of the subgrade. See H. M. Westergaard, Analytical Tools for Judging Results of Structural Tests of Concrete Pavements, *Public Roads*, Vol. 14, no. 10, p. 185, Dec. 1933.

variables E, p, h, k, and Γ, have been prepared for use in practical pavement analysis The large amount of pavement research that is now in progress will undoubtedly lead to new methods of pavement analysis, but it is to be expected that the equations that are developed will have the form of Equation a, provided that the concept of a subgrade modulus is retained

42 IMPACT

If a moving rigid body strikes a structure, the indentation depends on the size, the mass, and the velocity of the body On the other hand, the damage (bending or fracture) at some distance from the point of collision is practically independent of the size of the body, although it naturally depends on the mass of the body In any case, the length scale factor for a model of the incident body may be assumed to be the same as the length scale factor for a model of the structure Then, in a class of geometrically similar systems, the size of the structure and the size of the impinging body are both determined by a characteristic length I

The maximum stress σ at any point of the structure depends on the mass m and the velocity V of the incident body, the characteristic length L, the elastic modulus E, Poisson s ratio ν, and the mass density ρ of the structure According to Article 38 the constants E and ν characterize the material, even though yielding or fracture occurs However this reasoning does not take account of the effects of rate of loading The yield stresses of many materials are increased markedly if the loading is rapid Although, for expediency, we neglect the rate of strain, this simplification occasionally introduces appreciable scale effects in model studies of impact Consequently, the present analysis is inapplicable to very high speed phenomena, such as penetration of armor plates by projectiles

Assuming that the significant variables are L, m 1, E, ν ρ we obtain, by dimensional analysis,

$$\sigma = mV^2L^{-3}f\left(\frac{EL^3}{mV^2}, \nu, \frac{m}{\rho L^3}\right) \tag{a}$$

If a model and its prototype are made of the same materials, $K_E = K_\nu = K_\rho = 1$ Under these conditions, Equation a furnishes the following model law.

$$K_\sigma = 1, \quad K_V = 1, \quad K_m = K_L^3 \tag{b}$$

Equation b signifies that the model and the prototype experience the same stresses when they are struck by bodies moving with the same speed V, provided that the masses of the incident bodies are proportional to the cubes of the linear dimensions of the structures If damage is determined by stress, the two systems then experience the same damage For example, if two model ships collide, they experience the same damage as full-sized ships that collide at the *same speed as the models*, provided that the models and the prototypes are made of the same materials, and provided that geometric similarity is preserved with respect to all essential details. Geometric similarity implies that the masses of the models vary as the cubes of their linear dimensions.

As another example, suppose that a 1-oz stone, thrown with velocity V, will crack a certain window. Then the preceding model law implies that an 8-oz stone, thrown with the same velocity V, is required to crack a window whose width, height, and thickness are twice those of the first window. It should be recalled, however, that effects of rate of strain on stress may cause appreciable departures from results that are derived from Equation b. Also, there may be scale effects that do not depend on the rapidity of loading. The study of scale effects in impact tests of structures is a fruitful field for experimental research.

43. Small Free Vibrations of an Elastic System

It has been mentioned in Example 7 that a frictionless system may vibrate in such a manner that the particles execute simple harmonic motions that are in phase with each other. A vibration of this type is called a *natural mode*. To each natural mode, there corresponds a definite frequency (i.e. number of oscillations per second).

If attention is restricted to geometrically similar elastic systems, the frequency n of any specified natural mode depends on a length L that designates the size of the system, the mass density ρ of the material, Poisson's ratio ν, and Young's modulus E. The most general form of a dimensionally homogeneous equation among these variables is

$$n = \frac{k}{L}\sqrt{\frac{E}{\rho}} \tag{a}$$

in which k is a constant that may depend on ν. The resulting model law is

$$K_\nu = 1, \qquad K_n = \frac{1}{K_L}\sqrt{\frac{K_E}{K_\rho}} \tag{b}$$

In some cases, the condition $K_\nu = 1$ may be disregarded, since Poisson's ratio is irrelevant. If the model and the prototype are made of the same material, $K_E = K_\rho = 1$. Under these conditions, Equation b shows that the frequency of any natural mode is inversely proportional to the linear dimensions of the structure. For example, if the size of a tuning fork is doubled, the pitch is halved.

44 Small Forced Vibrations of an Elastic System

Suppose that the type of loading on an elastic structure is fixed, but that the magnitude of the load varies periodically with time, in a prescribed manner Then the force F that specifies the magnitude of the load on the structure is a periodic function of time Assuming the type of loading and the shape of the cycle of the force F to be given, we may specify the exciting forces completely by their frequency n and by a constant force F_0 (e g the maximum value of the force F)

The structure responds to the periodic load by vibrating Eventually the effects of the initial conditions are dissipated, and the subsequent vibration is periodic The amplitude A of the oscillation of any specified particle of the structure is then a function of the variables F_0, n, L, ρ, E, and ν, in which the notations are the same as in the preceding article Dimensional analysis now yields

$$A = \frac{F_0}{EL} f\left(\frac{F_0}{EL^2}, nL\sqrt{\frac{\rho}{E}}\right) \tag{a}$$

If the deflections are small, they are proportional to the applied load F_0 as in the case of statical loading Consequently, in the linear theory of vibrations, the term F_0/EL^2 drops out of Equation a Accordingly, the equation takes the simpler form,

$$A = \frac{F_0}{EL} f\left(nL\sqrt{\frac{\rho}{E}}\right) \tag{b}$$

In general, the form of Equation b is such that A becomes infinite for a certain value of n This value of n is known as the 'resonance frequency"

The model law that is derived from Equation b is

$$K_n^2 K_L^2 K_\rho = K_E, \qquad K_A = \frac{K_{F_0}}{K_E K_L} \tag{c}$$

If the model and the prototype are made of the same material, $K_E = K_\rho = 1$ Then, Equation c reduces to

$$K_n = \frac{1}{K_L}, \qquad K_{F_0} = K_A K_L \tag{d}$$

If the amplitudes of vibration of the model and the prototype are relatively the same, $K_A = K_L$ In this case, Equation d shows that the frequency of the exciting force should vary inversely as the size of the structure and that the magnitude of the exciting force should vary as the square of the size of the structure

For example, suppose that the lengths in a model of a steel bridge are one tenth of the corresponding lengths in the prototype and that the model is also made of steel. When a shaking force with a frequency of 30 cycles per second and a maximum value of 20 lb is applied to the model, certain vibrations are developed. To produce vibrations with corresponding amplitudes in the prototype, the frequency of the shaking force must be 3 cycles per second, and the maximum value of the shaking force must be 2000 lb.

PROBLEMS

1. Assuming that the stresses in a bridge are entirely due to the dead weight of the bridge, prove that, in a class of geometrically similar bridges, the larger bridges are more highly stressed. (Rayleigh)

2. A half-scale model of a cantilever plate-girder beam fails by buckling of the compression flange when the load is 40,000 lb. At what load will the prototype fail? How much will the prototype be deflected at the ultimate load if the maximum deflection of the model is 0.60 in.?

3. A bending moment of 5000 lb in. is required to bend a cylindrical steel bar into a U-shape. Another bar, made of the same steel as the first bar, has a diameter 50 percent greater than the first bar. What bending moment is required to bend the second bar into a U-shape?

4. Two identical steel balls are pressed together with sufficient force to cause a slight permanent flat spot on each ball. How does the force vary with the diameter of the balls?

5. Assuming that the stress-strain curve is of the form shown in Figure 4, plot to scale the graph of Figure 5 for beams of elliptical cross section.

6. The ultimate shearing stress for flat-head rivets is 34,000 $lb/in.^2$ (based on nominal rivet diameter), and the ultimate bearing stress of the sheet material for $e/d = 2$ is assumed to have the constant value, 100,000 $lb/in.^2$. Plot the graph of Figure 6 for this case.

7. The ratio of the ultimate bearing stress of rivets to the ultimate tensile stress of the sheet is 1.70. Plot the theoretical sheet-tear line and the theoretical crushing line for lap joints with three identical rows of rivets (see Figure 8). What is the ideal maximum joint efficiency for this case? Neglect stress concentration.

8. The ratio of the ultimate shearing stress of rivets to the ultimate tensile stress of the sheet is 0.50. The ratio of the rivet diameter to the sheet thickness is 2.50. Plot the theoretical sheet-tear line and the theoretical rivet-shear line for joints with two identical rows of rivets (see Figure 8). What is the ideal maximum joint efficiency for this case? Neglect stress concentration.

9. What is the ideal maximum efficiency for single-row lap joints, if the ratio of the ultimate bearing stress to the ultimate tensile stress of the sheet is 1.50? Neglect stress concentration.

10. Two identical steel balls experience a direct collision. Do the resulting stresses depend on the sizes of the balls? Explain.

11. The natural frequency of a steel tuning fork is 200 vibrations per second. What is the natural frequency of a 1/3 scale aluminum-alloy model of the tuning fork? For steel, $\rho = 15$ $slug/ft^3$, and $E = 30{,}000{,}000$ $lb/in.^2$. For aluminum alloy, $\rho = 5.4$ $slug/ft^3$ and $E = 10{,}500{,}000$ $lb/in.^2$.

12. Derive the model law for nonlinear forced vibrations of an elastic structure (see Equation a, Article 44). If $K_E = \frac{1}{2}$, $K_\rho = \frac{1}{3}$, and $K_L = \frac{1}{5}$, what are the values of K_A, K_{F_0}, and K_n?

13 If a beetle and a turtle are geometrically similar, which can fall from a greater height without injury? Explain your answer with and without the consideration of air resistance

14 The strain energy of an elastic plate, per unit area of the middle plane is expressed in terms of the deflection of the middle plane One part (membrane energy) is a homogeneous quadratic function of first derivatives of the deflection with respect to Cartesian coordinates in the middle plane Another part (bending energy) is a homogeneous quadratic function of second derivatives of the deflection and a third part (shear energy) is a homogeneous quadratic function of third derivatives of the deflection How do these energy terms vary with Young s modulus? How do they vary with the thickness of the plate?

15 Thin webbed beams used in aircraft are frequently stiffened by transverse ridges (beads) that are pressed into the webs The web shearing stress τ that will cripple the beads depends on the depth h of the web the spacing b between beads the depth d of a bead, the thickness t of the web the yield stress σ_y of the material, and Young s modulus E Make a dimensional analysis of the problem

CHAPTER 7

Some Applications of Dimensional Analysis in Fluid Mechanics

Dimensional analysis has played an important part in modern developments of fluid mechanics, particularly in investigations of turbulent flow. Some topics from the theory and practice of fluid mechanics, in which the reasoning is wholly or partly based on dimensional considerations, are presented in this chapter.

45. Velocity Distribution of Turbulent Flow in the Vicinity of a Solid Wall

Consider turbulent flow, in which the average stream lines are straight and parallel. This type of flow is exemplified by wind blowing over the plains, or by flow in a long straight pipe. The average velocity u at a distance y from the boundary depends on the roughness height e of the boundary, a length L that specifies the size of the system (e.g. the diameter of a pipe), the kinematic viscosity ν of the fluid, the mass density ρ of the fluid, and the shearing stress τ_0 that the fluid exerts on the boundary. Hence, there is a relationship of the type

$$f(u,\ y,\ e,\ L,\ \nu,\ \rho,\ \tau_0) = 0$$

Evidently, the ratios of lengths y/e and y/L are dimensionless products. Bearing this fact in mind, we may, for convenience, eliminate e and L from the dimensional matrix. Then the dimensional matrix is

	1	2	3	4	5
	u	y	ν	ρ	τ_0
M	0	0	0	1	1
L	1	1	2	−3	−1
T	−1	0	−1	0	−2

The rank of this matrix is three. Consequently, the five variables in the matrix furnish two independent dimensionless products. Proceeding as in previous examples, we obtain the dimensionless products,

$$u\sqrt{\frac{\rho}{\tau_0}} \quad \text{and} \quad \frac{y}{\nu}\sqrt{\frac{\tau_0}{\rho}}$$

The ratio $\sqrt{\tau_0/\rho}$ has the dimension of a velocity In the literature on fluid mechanics, it is called the *friction velocity*, and it is denoted by v^* Accordingly, a complete set of dimensionless products is

$$\frac{u}{v^*}, \quad \frac{yv^*}{\nu}, \quad \frac{y}{e}, \quad \frac{y}{L}$$

The product yv^*/ν is formally a Reynolds number This product is known as the *friction-distance parameter*

It now follows from Buckingham's theorem,

$$\frac{u}{v^*} = f\left(\frac{yv^*}{\nu}, \frac{y}{e}, \frac{y}{L}\right) \tag{19}$$

If the boundary is an infinite plane, there is no length L that characterizes the system, since the concept of ' size of the system" does not enter into consideration Consequently, in this case, the term y/L is dropped from Equation 19, i e , the equation takes the simpler form

$$\frac{u}{v^*} = f\left(\frac{yv^*}{\nu}, \frac{y}{e}\right) \tag{19'}$$

In any case, Equation 19 is valid for the region of high velocity gradient near a wall since, in this region, the ratio y/L is so small that it may be neglected Consequently, in all cases of uniform flow in a cylindrical or prismatic conduit, the velocity distribution near the wall is given by an equation of the form of Equation 19

The velocity fluctuations that characterize turbulence vanish at a smooth boundary Consequently very close to the boundary, the shearing stresses are primarily due to viscous action The region in which this condition prevails is called the *laminar sublayer* Although there are turbulent fluctuations at any finite distance from the wall, it is useful to conceive a definite thickness ϵ of the laminar sublayer Since ϵ depends on τ_0, ρ, and ν, we obtain, by dimensional analysis, the relationship,

$$\frac{\epsilon v^*}{\nu} = \text{constant} \tag{20}$$

Accordingly, the thickness of the laminar sublayer corresponds to a constant value of the friction distance parameter The value of the constant is rather indefinite, since it depends on an arbitrary designation of the thick ness of the laminar sublayer. According to the usual viewpoint, surface roughness has no effect on the flow if the surface irregularities are immersed in the laminar sublayer: i.e. if $e < \epsilon$. If this condition is adopted as a definition, the constant in Equation 20 is about 4.0, since Nikuradse[16] has shown by experiment that a surface behaves as though it were ideally smooth, if $ev^*/\nu < 4$. On the other hand, Nikuradse has shown that the viscosity has no perceptible influence on the velocity distribution near a wall, if the wall is so rough that $ev^*/\nu > 80$.

In the case of smooth surfaces ($ev^*/\nu < 4$), the term y/e may be dropped from Equation 19′. Consequently, the velocity distribution near a smooth surface is determined by an equation of the form,

$$\frac{u}{v^*} = f\left(\frac{yv^*}{\nu}\right) \tag{19''}$$

A complete formula for the function f is not known. However, Prandtl[16] deduced from his mixing-length hypothesis the following formula, which agrees closely with Nikuradse's experimental results for the range $yv^*/\nu >$ 50:

$$\frac{u}{v^*} = 5.75 \log_{10} \frac{yv^*}{\nu} + 5.5 \tag{21}$$

This equation was also derived by von Kármán by another method (Article 46).

Since, in the laminar sublayer, the shearing stress is primarily due to viscosity, Newton's equation for the shear is approximately correct for this region; i.e.,

$$\tau = \mu \frac{du}{dy}$$

The shearing stress throughout the laminar sublayer is practically equal to the wall shear τ_0. Accordingly, the preceding equation may be written:

$$(v^*)^2 = \nu \frac{du}{dy}$$

Integration yields

$$\frac{u}{v^*} = \frac{yv^*}{\nu} \tag{22}$$

This equation shows that the velocity distribution in the laminar sublayer is linear.

It is sometimes convenient to define the thickness ϵ of the laminar sub-

[16] L. Prandtl, *Strömungslehre*, Ch. III, Par. 5, F. Vieweg & Sohn, Braunschweig, 1949.

layer to be the value of y for which Equations 21 and 22 yield the same value of u/v^*. According to this definition, $ev^*/\nu = 11.8$. For values of the friction-distance parameter greater than 11.8, the Prandtl von Kármán Equation 21 is reasonably accurate, whereas Equation 22 may be used for values of the friction-distance parameter less than 11.8. Figure 9 is a graph of Equations 21 and 22.

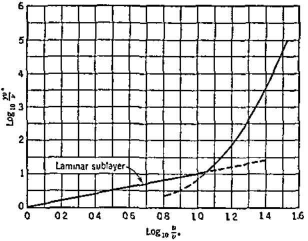

FIG. 9. Mean Turbulent Velocity Distribution near a Smooth Boundary (Equations 21 and 22)

In the case of a very rough surface ($ev^*/\nu > 80$) the viscosity term yv^*/ν may be dropped from Equation 19. Prandtl and von Karmán derived for this case, the equation

$$\frac{u}{v^*} = 5.75 \log_{10} \frac{y}{e} + 8.5 \tag{23}$$

This formula agrees with Nikuradse's experimental results for all values of y/e greater than 1.0.

46. Von Karman's Theory of the Shear in a Turbulent Velocity Field

In a turbulent velocity field the shearing stresses are usually several hundred times greater than in a comparable laminar flow. The great increase of shearing stress that accompanies turbulence is due to the momentum interchange caused by lateral fluctuations of velocity. Viscosity plays only a small part in this phenomenon.

Von Karman assumed that the shearing stresses at a point in a turbulent velocity field are determined by the density of the fluid and by the distribution of mean velocity in the neighborhood of the point. In a unidirectional flow, defined by $u = u(y)$, the distribution of velocity in the neighborhood of a point is determined by derivatives of u with respect to y. Accordingly, if only the first and second derivatives are significant, the relationship proposed by von Kármán is of the form,

$$\tau = f(\rho, u_y, u_{yy})$$

in which u_y and u_{yy} are abbreviations for du/dy and d^2u/dy^2. This relationship admits only one dimensionally homogeneous form; namely,

$$\tau = \frac{\rho \kappa^2 u_y^4}{u_{yy}^2} \tag{24}$$

in which κ is a dimensionless constant. If the shear due to viscosity is relatively small, it may be taken into account by adding the Newtonian friction term to Equation 24. Then,

$$\tau = \mu u_y + \frac{\rho \kappa^2 u_y^4}{u_{yy}^2} \tag{25}$$

This is von Kármán's equation. Experiments indicate[17] that the value of κ is about 0.40.

If τ is a known function of y, Equation 25 may be integrated. The simplest case is that in which the plane $y = 0$ is a smooth boundary. Then, since there is a negligible pressure gradient, the shearing stress τ is constant. This follows from the equilibrium equations. Von Kármán assumed that the term μu_y in Equation 25 has a negligible influence on the flow in the region outside of the laminar sublayer. Consequently, he discarded this term. However, the integral of the equation is then not dimensionally homogeneous, unless the viscosity is reintroduced in one of the constants of integration, since Equation 19″ is the only dimensionally homogeneous form for the solution. A procedure that is mathematically more satisfactory is to leave the viscosity term in Equation 25 and to solve the differential equation subject to the boundary conditions $u = 0$ and $\tau = \mu u_y$ for $y = 0$. However, the resulting velocity distribution does not agree with experimental results. There are several possible explanations for the discrepancy. In the first place, Equation 25 is not the most general dimensionally homogeneous equation that expresses τ as a function of the four variables μ, ρ, u_y, and u_{yy}. In the second place, higher derivatives of u are probably significant. However, the influence of higher derivatives, and the perturbations due to viscosity effects, other than those indicated

[17] H. Rouse, *Fluid Mechanics for Hydraulic Engineers*, Ch. XII, McGraw-Hill, New York, 1938.

by Equation 25, are both small in the region outside of the laminar sub layer Since it is recognized that Equation 25 is acceptable only for the region outside of the laminar sublayer, the constants of integration may not be determined by the boundary conditions Consequently, these constants must be obtained empirically

The details of the calculation are now easily performed Equation 25 is approximated by Equation 24, which may be written,

$$v^* = -\frac{\kappa u_y^2}{u_{yy}}$$

The first integration yields $v^* = \kappa y u_y$, in which the constant of integration is set equal to zero The second integration yields

$$v^* \ln y = \kappa u + C \tag{a}$$

To obtain a dimensionally homogeneous form, set $C = C' - v^* \ln (v^*/\nu)$ Then,

$$v^* \ln \frac{yv^*}{\nu} = \kappa u + C'$$

This agrees with Equation 21, if $\kappa = 0\,40$ and $C' = -5\,5\kappa v^*$

Equation a also furnishes the solution for very rough walls ($\epsilon v^*/\nu > 80$) In this case, the viscosity has no effect but the roughness height ϵ appears in the equation To obtain a dimensionally homogeneous equation, set $C = C' + v^* \ln \epsilon$ Then Equation a takes the following form

$$v^* \ln \frac{y}{\epsilon} = \kappa u + C'$$

This agrees with Equation 23, if $\kappa = 0\,40$ and $C' = -8\,5\kappa v^*$

For flow in a pipe, τ is a linear function of y Von Kármán integrated Equation 24 for this case Prandtl has pointed out, however, that the linear variation of τ in a pipe has little influence on the velocity profile, since the region of high velocity gradient is near the wall Actually, Equations 21 and 23 are even more accurate than von Kármán's solution for the entire velocity profile of flow in a pipe However, Equations 21 and 23 are inaccurate if Reynolds' number is in the range 2000 to 100,000, since viscosity exerts a strong effect in this range Here, only empirical relationships are known

Another form of the equation that is applicable to uniform flow in a pipe is obtained by setting $C = C' + v^* \ln r$ in Equation a, in which r is the radius of the pipe Then Equation a becomes

$$v^* \ln \frac{y}{r} = \kappa u + C' \tag{b}$$

Now, for $y = r$, $u = u_{\max}$. Consequently, Equation b yields

$$\frac{u_{\max} - u}{v^*} = \frac{1}{\kappa} \ln \frac{r}{y} \tag{26}$$

This equation is equally valid for rough pipes and for smooth pipes. It may be seen from Equation 26 that the shape of the central part of the velocity profile does not depend directly on the wall roughness nor on the

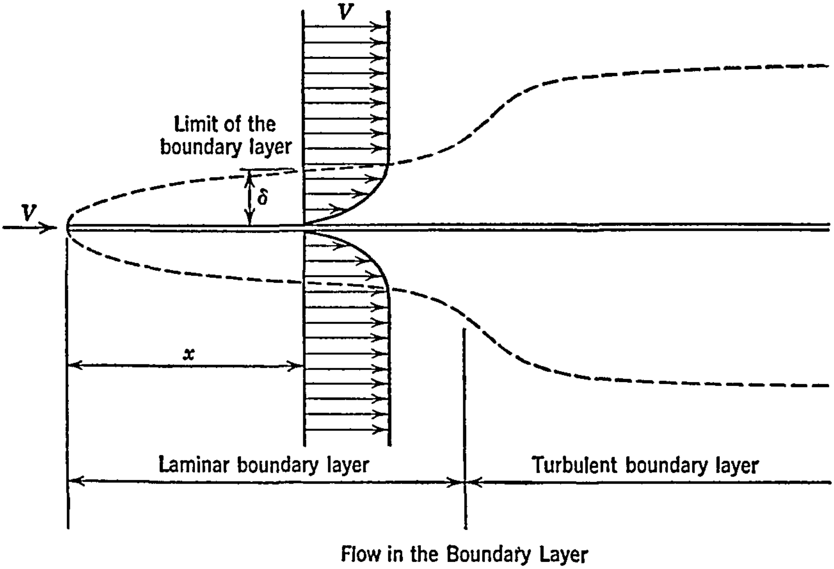

FIG. 10. Flow in the Boundary Layer

average velocity in the pipe, but only on the wall shearing stress τ_0. According to Equation 26, there is a slight cusp at the center of the velocity profile. This discrepancy appears also in von Kármán's solution.

47. Boundary-Layer Theory

If a body is immersed in a stream of fluid, the drag force on the body is caused partly by differences of pressure on the front and the rear parts of the body, and partly by shearing stresses on the surface of the body. The portion of the drag force that results from the shearing stresses is called "skin-friction drag." If a very thin plate is placed edgewise to a stream of fluid, the drag is entirely attributable to skin friction. The nature of the flow, in this case, is indicated by Figure 10. The plate causes very little disturbance of the velocity field in the region ahead of its leading edge. However, near the surface of the plate, there is a high velocity gradient. The region in which an appreciable velocity gradient exists is called the *boundary layer* Evidently, the thickness δ of the boundary layer is an increasing function of the coordinate x (Figure 10)

The shearing stress τ_0 on the surface of the plate may be represented in the form,

$$\tau_0 = \tfrac{1}{2}c_f \rho V^2 \tag{a}$$

The friction coefficient c_f is evidently dimensionless Frequently, the average friction coefficient for the interval $(0, x)$ is used This coefficient is naturally defined by the equation,

$$C_f = \frac{1}{x}\int_0^x c_f\,dx \tag{b}$$

The coefficient C_f may be directly determined by experiment, whereas the coefficient c_f cannot be directly measured The drag force on one side of the plate in the interval $(0\ x)$ is for a unit width

$$F = \tfrac{1}{2}C_f \rho V^2 x$$

The boundary layer thickness and the friction coefficients are functions of the coordinate x the free stream velocity V the kinematic viscosity ν and the roughness height e of the surface Dimensional analysis yields

$$\delta = xf\left(\frac{Vx}{\nu}, \frac{Ve}{\nu}\right) \tag{c}$$

$$c_f = f_1\left(\frac{Vx}{\nu}, \frac{Ve}{\nu}\right) \tag{d}$$

$$C_f = f_2\left(\frac{Vx}{\nu}, \frac{Ve}{\nu}\right) \tag{e}$$

The dimensionless product Vx/ν is Reynolds number It is denoted by $\mathbf{R}_x$ The dimensionless product $\mathbf{R}_\delta = V\delta/\nu$ is also employed in the literature on the boundary layer

At some distance downstream from the leading edge of the plate the flow in the boundary layer undergoes a transition from laminar motion to turbulence Reasoning similar to that in Example 6 shows that the transition occurs at a critical value of Reynolds number However the critical Reynolds number is found to be strongly influenced by surface roughness and by conditions at the leading edge of the plate Usually the transition occurs in the range $10^5 < \mathbf{R}_x < 10^6$ In terms of $\mathbf{R}_\delta$ this range is roughly $1600 < \mathbf{R}_\delta < 5000$

The velocity distribution in the laminar boundary layer does not depend appreciably on the density since the inertia forces are slight Accordingly, dimensional analysis yields

$$u = Vf\left(\frac{y}{\delta}\right) \tag{f}$$

in which u is the velocity at a distance y from the boundary. Note that the viscosity is excluded from this equation on dimensional grounds. Equation f means that the lines $y = k\delta$ (i.e., lines at constant percentages of the depth of the boundary layer) are lines of constant velocity.

Von Kármán[18] employed Equation f in conjunction with the momentum equation,

$$F = \rho \int_0^\delta u(V - u)\, dy \tag{g}$$

in which F is the drag force (per unit width) on one side of the plate in the interval $(0, x)$. This equation is derived in most books on fluid mechanics.

Since $dF/dx = \tau_0$, Equation g yields

$$(v^*)^2 = \frac{d}{dx} \int_0^\delta u(V - u)\, dy \tag{h}$$

With Equation f, this yields

$$(v^*)^2 = \alpha V^2 \frac{d\delta}{dx} \tag{k}$$

in which α is a constant.

Also, the shearing stress at the boundary is given by $\tau_0 = \mu(du/dy)_0$. With Equation f, this yields

$$(v^*)^2 = \frac{\beta \nu V}{\delta} \tag{m}$$

in which β is a constant. Equations k and m provide a simple differential equation for δ. The solution is

$$\delta = \sqrt{\frac{2\nu\beta x}{\alpha V}} \tag{n}$$

Equations m and n yield

$$c_f = \sqrt{\frac{2\alpha\beta\nu}{Vx}} \tag{p}$$

The constants α and β may be obtained from the assumption that the velocity distribution in the boundary layer is parabolic, as for flow between parallel plane walls. For greater accuracy, von Kármán recommended a velocity profile in the form of a cubic parabola. However, Blasius' calculation of these constants, based on the differential equations of viscous flow, is more rigorous. With Blasius' constants, Equations n and p may be expressed

$$\delta = 5.2x\mathrm{R}_x^{-1/2}, \qquad c_f = 0.664\mathrm{R}_x^{-1/2} \tag{q}$$

[18] T. von Kármán, Turbulence and Skin Friction, *J. Aeronaut. Sci.*, Vol. 1, no. 1, 1934. See also: Mechanische Ähnlichkeit und Turbulenz, *3d Inter. Congr. Applied Mechanics*, Vol. 1, p. 84, Stockholm, 1930.

Von Kármán assumed that the velocity distribution in the turbulent boundary layer over a smooth plate is given by Equation 21. Since $u = V$ for $y = \delta$, Equation 21 yields

$$\frac{V}{v^*} = \frac{1}{\kappa} \ln \frac{\delta v^*}{\nu} + 5.5 \tag{r}$$

If the velocity distribution of Figure 9 is substituted in the momentum equation h, the integration may be performed. The boundary-layer thickness δ may be eliminated from the integral by Equation r. Thus, a differen-

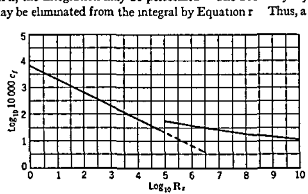

FIG. 11. Friction Coefficient for Smooth Plates

tial equation is obtained for the friction coefficient c_f. Von Kármán integrated* this equation. For large Reynolds numbers, his solution is approximated by

$$\kappa \sqrt{\frac{2}{c_f}} = \ln (R_x c_f) + 1.15 \tag{s}$$

Equation s is cumbersome to use, since it is not an explicit equation for c_f. Schlichting proposed the following explicit formula for the average friction coefficient C_f of the turbulent boundary layer

$$C_f = 0.455(\log_{10} R_x)^{-2.58} \tag{t}$$

In view of Equation b, the local friction coefficient c_f is derived from the average friction coefficient C_f by the equation,

$$\frac{d}{dx}(xC_f) = c_f \quad \text{or} \quad \frac{d}{dR_x}(R_x C_f) = c_f$$

* Instead of Figure 9, von Kármán used the velocity distribution that is derived from Equation 24 by assuming a linear distribution of shearing stress.

Hence, by Equation t,

$$c_f = 0.455(\log_{10}R_x)^{-2.58} - 0.510(\log_{10}R_x)^{-3.58} \tag{u}$$

Equation u agrees closely with von Kármán's result. It also agrees closely with experimental measurements of the boundary layer at large Reynolds numbers. Figure 11 is a graph of the friction coefficient c_f, based on Equations q and u. This graph applies only for surfaces that are practically smooth. When c_f is known, δ may be obtained from Equation r.

The foregoing theory is used for calculating the skin-frictional resistances of boats and airplanes, since, insofar as skin friction is concerned, the surfaces of these bodies may usually be approximated by flat plates. Equation t is more convenient to apply than Figure 11, since it determines the total skin-friction drag without integration. However, Equation t is inapplicable if a large part of the leading portion of the boundary layer is laminar. Usually, the Reynolds numbers of full-scale boat hulls are sufficiently large to permit the use of Equation t, but this equation is frequently inaccurate for small-scale models.

48. Uniform Flow in a Flume or a Conduit

The "wetted perimeter" of the cross section of a stream is that part of the perimeter that is contiguous with the walls. The size of the cross section of a stream in a flume or in a closed conduit is frequently designated by the *hydraulic radius R*, which is defined to be the ratio of the cross-sectional area of the stream to the wetted perimeter. In the literature on hydraulics, criticisms have been directed at the frequent assumption that the rate of flow in a flume is determined by the hydraulic radius alone, without regard for the shape of the cross section. However, there can be no logical objection to the specifying of the size of the cross section of a stream by the hydraulic radius, if the shape of the cross section is recognized to have an effect. A difficulty in the testing of open channels arises because the shape of the cross section of a stream is generally changed when the depth of the stream is changed. Flumes of triangular cross section are an exception to this remark.

If the flow in a conduit (either open or closed) is uniform, Bernoulli's equation, in conjunction with the Darcy formula for the energy loss (Equation 3), yields

$$s = \frac{f}{4R}\frac{V^2}{2g} \tag{27}$$

in which s is the loss of piezometric head per unit length (i.e. the slope of the hydraulic grade line). For an open channel, s is identical to the bottom slope. The term f is the friction factor, defined in Example 5.

Equation 27 shows that the mean velocity V in a prismatic or cylindrical conduit depends on the hydraulic radius R, the kinematic viscosity ν, the roughness height ϵ, the product gs, and the shape of the cross section, i e,

$$V = f(gs, R, \nu, \epsilon, \text{shape})$$

By dimensional analysis, this equation is reduced to the following form:

$$V = \sqrt{gRs}\, F\left(\frac{R}{\nu}\sqrt{gRs}, \frac{\epsilon}{R}, \text{shape}\right) \tag{28}$$

If $\sqrt{g}\,F$ is identified as the Chézy coefficient C, Equation 28 is the Chézy formula, $V = C\sqrt{Rs}$, which is the basis of much of the engineering theory

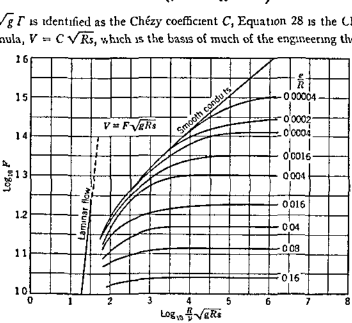

FIG 12 Inversion of Stanton Diagram

of flow in open channels Several empirical formulas for the coefficient C are in general use However, none of these formulas is dimensionally homogeneous Experimental charting of the function F is much needed

For pipes, the function F may be obtained by inversion of the Stanton diagram The procedure is as follows

Choose a value of Reynolds' number ($\mathrm{R} = 4VR/\nu$), and obtain the corresponding value of f from the Stanton diagram (Figure 2) By Equations 27 and 28, $F = \sqrt{8/f}$ Hence, F may be computed Also, the dimensionless product $(R/\nu)\sqrt{gRs}$, on which F depends, is obtained by the identity,

$$\frac{R}{\nu}\sqrt{gRs} = \frac{VR}{\nu}\frac{\sqrt{gRs}}{V} = \frac{VR}{\nu F}$$

Thus, from the Stanton diagram, a chart with ordinate F and abscissa $R/\nu\sqrt{gRs}$ is obtained. The parameter of the family of curves is e/R. Figure 12 is the chart that is derived from Figure 2.

The experience of hydraulicists indicates that the cross-sectional shape of a flume has little effect on the flow if the shearing stress is nearly constant on the wetted perimeter. Consequently, Figure 12 may be expected to apply for open channels with simple cross-sectional shapes. Also, Figure 12 is more convenient than the Stanton diagram for solving pipe-line problems in which the gradient of piezometric head is known and the velocity is to be determined.

Experiments on open channels at the University of Illinois indicate that, at high slopes, the dimensionless variable s exerts an influence that is independent of the product gs. Probably wave action accounts for this discrepancy, since a small amount of wave action may dissipate as much energy as the wall friction. If waves exist, the flow is not truly uniform.

49. Run-off From a Watershed

While in the employ of the Division of Waterways of the Illinois Department of Public Works and Buildings, Mr. Cevdet A. Erzen made a useful application of dimensional analysis to a problem of hydrology. The author is indebted to Mr. Erzen and to the Illinois Division of Waterways for the use of this example.

The watershed of a river is the territory that is drained by the river. If a rain falls on a watershed, the river rises. However, the maximum stage of the river does not occur immediately after the rain. The discharge of the river may continue to increase for a number of days after the rainfall, since time is required for the water to drain from the watershed into the river. The problem then arises of tracing the history of water after it falls as rain. At a time t after the rainfall, an amount Q (ft^3/sec) of the rainwater is being discharged by the river. Specifically, the problem is to express Q as a function of t. The effect of the duration of the rainstorm on the discharge Q will be neglected. Mathematically, this is tantamount to the assumption that all the rain falls at the initial instant, $t = 0$. The amount of rainfall H is usually expressed as "feet of rain." This is the total rainfall on the watershed (ft^3) divided by the area of the watershed (ft^2).

The run-off Q from a watershed naturally depends on the topography, the vegetation, the nature of the soil, and the initial percent of saturation of the soil. It is also strongly influenced by snow and ice. Since it is difficult to define these variables, attention must be restricted to watersheds that are physically similar. Then the most important variables that influence the run-off Q at the time t are the amount of rainfall H, the area of the watershed A, the acceleration of gravity g the mass density ρ of water, and the kinematic viscosity ν of water The viscosity ν depends on the temperature

The relationship among the variables is indicated by

$$f(Q, t, A, H, g, \rho, \nu) = 0$$

It is impossible to form a dimensionless product that contains ρ, since ρ is the only variable that contains the dimension of mass Therefore, ρ actually does not enter the problem

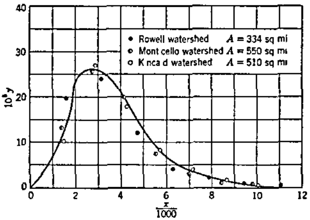

FIG 13 Dimensionless Graph Showing Run offs from Three Watersheds in Illinois By courtesy of the Illinois Division of Waterways

Dimensional analysis of the above equation yields

$$Q = \sqrt{gH^5}\, f\left(t\sqrt[4]{\frac{g^2}{A}}, \frac{A}{H^2}, \frac{\nu^2}{gA^{3/2}}\right) \tag{a}$$

It is known from observation that Q is approximately proportional to H Therefore, Equation a must take the more special form

$$Q = g^{1/2}A^{3/4}Hf(tg^{1/2}A^{-1/4}, \nu^2 g^{-1}A^{-3/2}) \tag{b}$$

For brevity, let us set

$$x = tg^{1/2}A^{-1/4}, \qquad y = Qg^{-1/2}A^{-3/4}H^{-1}, \qquad k = \nu^2 g^{-1}A^{-3/2}$$

Then Equation b becomes

$$y = f(x, k) \tag{c}$$

The foregoing argument does not take evaporation into account It may be possible to include this effect in the dimensional analysis of the problem, but, for simplicity, evaporation has been disregarded. Accordingly, all of the rainwater is considered to run off in the river. Hence,

$$\int_0^\infty Q\,dt = AH$$

In terms of the variables x and y, this equation is expressed,

$$\int_0^\infty y\,dx = 1 \tag{d}$$

Consequently, if a curve is plotted with ordinate y and abscissa x, the area under the curve is unity. In practice, the upper limit of the integral may be replaced by a reasonable finite value.

The graph of Equation c should be expected to be approximately the same for all watersheds that are geologically similar and that have roughly the same shape. Data of the Illinois Division of Waterways tend to confirm this conjecture. Figure 13 shows data from three watersheds in the state of Illinois. Note that all points fall practically on the same curve. Although different curves should be expected for different values of the parameter k, the data are not extensive enough to exhibit this effect.

50. Centrifugal Pumps

In a class of geometrically similar centrifugal pumps, a pump is specified by a single length—say, the diameter D of the impeller. A manufacturer usually builds several classes of centrifugal pumps, such that pumps in the same class are approximately geometrically similar. The purchaser of a pump specifies the head and the rate of flow Q at which the pump is to operate. The head may be defined as the difference $P = (p_2 + \rho g Z_2) - (p_1 + \rho g Z_1)$, in which the subscripts 1 and 2 refer to the suction side and the discharge side of the pump. After the type of pump is selected, the size of the pump to furnish the specified values of P and Q is determined by an equation of the form,

$$D = f(Q, P, \rho, n)$$

in which n is the angular speed (rev/sec) of the shaft. Viscosity ordinarily exerts only a secondary effect, and it is consequently not included in this discussion. Dimensional analysis of the above equation yields

$$\frac{D^3 n}{Q} = f_1\left(n\sqrt[4]{\frac{Q^2\rho^3}{P^3}}\right) \tag{a}$$

The dimensionless product $n\sqrt[4]{Q^2\rho^3/P^3}$ is known as *specific speed.*

The speed of a pump is usually determined by the speeds of available motors. Consequently, the purchaser's specifications largely determine the specific speed of a pump The diameter D of the impeller (i e the size of the pump) is then determined by Equation a The manufacturer's experimental data provide a graph of the function f_1

The efficiency η of pumps of a given class is also a function of specific speed, i e ,

$$\eta = f_2\left(n\sqrt[4]{\frac{Q^2\rho^3}{P^2}}\right) \tag{b}$$

If the class of pumps is appropriately selected, the specific speed at which the pump is to operate is near the specific speed for maximum efficiency

Cavitation (i e boiling caused by low pressure) is frequently encountered in hydraulic machines Cavitation naturally depends on the pressure on the suction side of the pump This, in turn, is determined by the static pressure at the shaft level, i e , $p_a - \rho g h$, in which p_a is the atmospheric pressure and h is the height of the center line of the pump above the water level in the sump Since only the excess of pressure above the vapor pressure p_v of the liquid is significant, the pressure that is directly related to cavitation is $p = p_a - p_v - \rho g h$

The rotational speed n_c at which a particular pump will cavitate is given by an equation of the form,

$$n_c = f(\rho, P, p \; D)$$

Dimensional analysis of this equation yields

$$n_c = \sqrt{\frac{P}{\rho D}}\, f\left(\frac{p}{P}\right) \tag{c}$$

The ratio p/P is called the Thoma number " When the function $f(p/P)$ is known for a class of pumps Equation c determines whether or not a certain pump will cavitate under specified operating conditions Cavitation impairs efficiency and it is injurious to the machine

In the foregoing discussion the performance of centrifugal pumps has been considered from the viewpoint of the engineer who must choose a pump to fulfill specified conditions From another viewpoint, P may be considered to be a function of the four variables Q n, D, ρ, for a certain class of pumps, i e ,

$$P = f(Q, n, D, \rho)$$

By dimensional analysis this equation is reduced to the form,

$$\frac{P}{\rho n^2 D^2} = f_1\left(\frac{Q}{nD^3}\right) \tag{d}$$

Likewise

$$\frac{E}{\rho n^3 D^5} = f_2\left(\frac{Q}{nD^3}\right) \tag{e}$$

and
$$\eta = f_3\left(\frac{Q}{nD^3}\right) \tag{f}$$

Here E denotes the power (ft lb/sec) required to drive the pump. The dimensionless product $P/(\rho n^2 D^2)$ is a form of the pressure coefficient. The dimensionless product $E/(\rho n^3 D^5)$ is known as the "power coefficient."

The function f_3 cannot be a constant, since $\eta = 0$ if $Q = 0$. Consequently, Equation f shows that, if η is constant, the ratio Q/nD^3 is constant; i.e., $Q = KnD^3$, in which K is a constant. This conclusion may be expressed as follows:

> When a centrifugal pump operates at constant efficiency, the rate of discharge Q is proportional to the speed n, and proportional to the cube of the diameter of the impeller. The discharge Q does not depend on the density of the liquid.

Since $f_1(K)$ is constant, Equation d now yields the following principle:

> When a centrifugal pump operates at constant efficiency, the head P is proportional to the density of the liquid, proportional to the square of the rotational speed, and proportional to the square of the diameter of the impeller.

Finally, since $f_2(K)$ is constant, Equation e yields the principle:

> When a centrifugal pump operates at constant efficiency, the shaft horsepower (or the water horsepower) is proportional to the density of the liquid, proportional to the cube of the rotational speed, and proportional to the fifth power of the diameter of the impeller.

Since the maximum efficiency of a pump is approximately independent of ρ, n, and D, the preceding principles are valid for the case in which the pumps are operated at maximum efficiency.

The equations show that the power coefficient, the pressure coefficient, and the efficiency of a centrifugal pump may be plotted as functions of the dimensionless variable Q/nD^3. When test data are given in this form, it is unnecessary to specify the speed of rotation, the density of the liquid, or the size of the pump. Of course, the data are valid only for pumps that are geometrically similar.

51. Centrifugal Compressors

The flow of a gas is influenced by the ratio of specific heats C_p/C_v. This constant is approximately 1.40 for diatomic gases. In any case, the ratio of specific heats may be disregarded if attention is restricted to a single gas. Also, the acceleration of gravity does not ordinarily appear in formulas dealing with the flow of gases, since the weight of a gas is usually negligible compared to the other forces that act on it

In view of these remarks, the pressure p on the discharge side of a centrifugal compressor is determined by the pressure p_0 on the inlet side, the mass density ρ_0 of air on the inlet side, the mass m of air that flows through the machine per second, the speed n (rev/sec) of the machine, and the diameter D of the rotor The specification of a machine by the diameter D implies that a class of geometrically similar machines is considered If the machines are water jacketed, the calculation of performance is partly a problem of heat transfer, and a number of other variables must be considered Restricting attention to the simpler case, we have

$$p = f(p_0, \rho_0, m, n, D)$$

Dimensional analysis of this equation yields

$$p = p_0 f\left(\frac{m}{nD^3\rho_0}, \frac{p_0}{n^2D^2\rho_0}\right) \tag{a}$$

Now, $m/\rho_0 = Q$, the volume of air at inlet conditions that flows through the machine per second Also the speed of sound at inlet conditions is given by the equation $c_0 = \sqrt{1\ 4(p_0/\rho_0)}$ Consequently Equation a may be written

$$\frac{p}{p_0} = f_1\left(\frac{Q}{nD^3}, \frac{c_0}{nD}\right) \tag{b}$$

Similarly, the power coefficient $E/\rho_0 n^3 D^5$ and the efficiency η are given by equations of the type,

$$\frac{E}{\rho_0 n^3 D^5} = f_2\left(\frac{Q}{nD^3}, \frac{c_0}{nD}\right) \tag{c}$$

$$\eta = f_3\left(\frac{Q}{nD^3}, \frac{c_0}{nD}\right) \tag{d}$$

Here E is to be regarded as the shaft power (ft lb/sec) minus the bearing losses

In view of the gas law $p_0 = \rho_0 R\theta_0$ the speed of sound is determined by the temperature alone Consequently Equation b shows that, when Q, n, and θ_0 are constant, p is proportional to p_0

The problem of choosing a centrifugal compressor to do a specified job is similar to the problem of choosing a pump If a certain class of machines is selected, the diameter D is the only unknown Regarding D as a dependent variable, we obtain, by dimensional analysis,

$$D = \frac{c_0}{n} f\left(n\sqrt{\frac{Q}{c_0^3}}, \frac{p}{p_0}\right) \tag{e}$$

The efficiency η is also given by an equation of the form,

$$\eta = F\left(n\sqrt{\frac{Q}{c_0^3}}, \frac{p}{p_0}\right) \qquad \text{(f)}$$

The functions f and F may be obtained experimentally.

The dimensionless product $n\sqrt{Q/c_0^3}$ is known as *specific speed.* If p/p_0 is specified, the specific speed may be chosen to furnish maximum efficiency. Then the size of the machine is determined by Equation e. If the speed n that is obtained by these calculations is impracticable, the proposed class of machines is unsuitable.

For a more complete treatment of the applications of dimensional analysis to problems of turbomachines, the reader is referred to books on turbomachinery.[19,20]

Problems

1. A steady wind blows over plowed fields in level country. At a height of 10 ft above the ground, the wind velocity is 40 ft/sec. The roughness height of the fields is 0.5 ft ($\rho = 0.00238$ slug/ft^3 and $\nu = 0.000156$ ft^2/sec). Calculate the friction velocity. Calculate the shearing stress of the wind on the ground. Calculate the wind velocity at an altitude of 1 ft. At an altitude of 100 ft. Plot the velocity profile.

2. Solve Problem 1 for the case in which the wind is blowing over a large smooth frozen lake. What is the thickness of the laminar sublayer?

3. Determine the most general form of a dimensionally homogeneous equation for the shearing stress τ in a unidirectional turbulent velocity field, if it depends on the mass density ρ, the kinematic viscosity ν, and the three derivatives u_y, u_{yy}, and u_{yyy}.

4. Liquid flows uniformly in an open channel. Prove that the average friction velocity is $\sqrt{gRs}$.

5. Water ($\nu = 1.1 \times 10^{-5}$ ft^2/sec) flows through a horizontal pipe 1 ft in diameter. The roughness height is 0.002 ft. The pressure drop is 3 lb/ft^2 per foot of length. Using Figure 12, calculate the average velocity in the pipe. Calculate Reynolds' number. Calculate the friction velocity. Calculate the velocity at the center line of the pipe.

6. Water flows through a pipe 6 in. in diameter. The velocity at the center line of the pipe is 20 ft/sec. The friction on the wall of the pipe is 1.5 lb/ft^2. Calculate the velocity u at the points $y/r = 0.05, 0.10, 0.20, 0.40, 0.60, 0.80$. Plot the velocity profile.

7. A centrifugal pump with an impeller 12 in. in diameter delivers 400 gal/min of water when turning 1800 rev/min. A geometrically similar pump with an impeller 24 in. in diameter turns 1200 rev/min, and it has the same specific speed as the first pump. What is the rate of discharge of the larger pump, if it is pumping kerosene with specific gravity 0.82? What is the ratio of the heads P of the two pumps?

8. The maximum rate of discharge Q_{max} of a centrifugal pump is reached when violent cavitation occurs. In a class of geometrically similar pumps, Q_{max} depends on the diameter D of the impeller, the speed of rotation n, the density ρ of the liquid, and the

[19] G. Wislicenus, *Fluid Mechanics of Turbomachinery*, Chs. 2, 3, and 4, McGraw-Hill, New York, 1947.

[20] W. Spannhake, *Centrifugal Pumps, Turbines, and Propellers*, Technology Press, Massachusetts Institute of Technology, Cambridge, Mass., 1934.

cavitation pressure function p Derive dimensionless coordinates of a single curve that determines Q_{max} for all combinations of D, n, ρ and p

9 In a class of geometrically similar machines the run away speed n of a hydraulic turbine depends on the diameter D of the runner, the head P the mass density ρ of water the time t after run away starts, and the initial speed n_0. Derive the most general form of a dimensionally homogeneous equation for the run away speed n

10 From a class of geometrically similar hydraulic turbines a designer wishes to select a turbine that will deliver specified power E at a specified speed n with a specified value of the head P What is the general form of the equation that determines the diameter D of the runner? What is the general form of the equation that determines the quantity Q of water that flows through the turbine per second? What is the general form of the equation for the efficiency? What dimensionless product may be identified as specific speed in this case?

11 Two geometrically similar centrifugal air compressors operate in the same room One compressor is twice as large as the other The larger compressor runs at half the speed of the smaller one Both machines operate at maximum efficiency How do the mass rates of discharge of the two machines compare with each other? How do the discharge pressures compare? How do the power requirements compare?

12 Derive Equation 24 by the method of Chapter 3

13 Prove that in a class of geometrically similar centrifugal pumps the specific speed at which cavitation occurs is a function of the Thoma number

14 Derive the formula for C_f for the laminar boundary layer

15 For the calculation of skin friction drag the submerged part of the hull of a ship is represented by a thin rectangular flat plate 30 ft high and 500 ft long If the speed of the plate is 30 ft/sec and the critical Reynolds number is 300 000 how far aft of the leading edge of the plate does the boundary layer get turbulent? What is the total drag force on the plate? How much horsepower does the ship expend in overcoming skin friction? ($\nu = 1\ 41 \times 10^{-5}$ ft^2/sec $\rho = 2\ 0$ slug/ft^3)

16 Using Erzen's chart determine the maximum rate of run off from a watershed 1000 sq mi in area if the precipitation is 6 in of rain How many days after the rainfall does the maximum run off occur?

17 List the variables that determine the speed of rotation n_c at which the propeller of a boat will cavitate Make a dimensional analysis of the problem Derive the model law

18 A strut of given shape is towed through water Prove by dimensional analysis that the velocity at which cavitation begins does not depend on the size of the cross section of the strut

19 Water from a stand pipe flows under a sluice gate into a horizontal flume that is connected to the bottom of the stand pipe Prove by dimensional analysis that the depth of water at the vena contracta does not vary as the head in the stand pipe reduces

20 If a localized disturbance moves at constant speed in a straight line on the surface of deep water, the gravity waves that are created by the disturbance are confined to a wedge shaped region with the point of disturbance at its apex Prove that the angle of the wedge does not depend on the speed of the point of disturbance.

CHAPTER 8

Dimensional Analysis Applied to the Theory of Heat

52. Dimensions of Thermal Entities

In his book, *Theory of Heat*, Clerk Maxwell aptly remarked that special units of heat, such as the British thermal unit or the calorie, are needless encumbrances to the science of heat. Indeed, there is no other branch of physics that employs a special unit of energy. In electromagnetic theory, mechanical units of energy (e.g. the erg or the joule) are consistently used. As Maxwell has pointed out, special units of heat lead to awkward phraseologies in scientific discussions, and they cause irregular outcroppings of conversion factors in the equations of thermodynamics. Accordingly, for simplicity, we shall consider heat to be expressed in mechanical units (joules, foot pounds, etc.). A quantity of heat then has the dimension of work; i.e. $[FL]$ or $[ML^2T^{-2}]$.

Since the thermometric scale is independent of the definitions of mechanical units, a new fundamental dimension $[\Theta]$ is assigned to temperature. Specific heat, being the quantity of heat that is required to raise a unit mass one degree of temperature, then has the dimension $[L^2T^{-2}\Theta^{-1}]$. The heat capacity per unit volume (i.e. the quantity of heat that is required to raise a unit volume of material one degree of temperature) is $C\rho$, in which C is the specific heat and ρ is the mass density.

The extension of a rod of length L, due to a temperature rise $\Delta\theta$, is $\beta L\Delta\theta$. Consequently, the dimension of the coefficient of thermal expansion β is $[\Theta^{-1}]$.

If an insulated prismatic bar of cross-sectional area A and length L has its ends maintained at temperatures θ_1 and θ_2, the quantity of heat that flows per unit time through any cross section is $k(A/L)(\theta_1 - \theta_2)$, in which k is the coefficient of thermal conductivity. Consequently, the dimension of the coefficient of thermal conductivity is $[FT^{-1}\Theta^{-1}]$ or $[MLT^{-3}\Theta^{-1}]$.

If a hot plate is immersed in a fluid, the quantity of heat that is transferred from the plate to the fluid per unit time is $hA\,\Delta\theta$, in which A is the area of the plate and $\Delta\theta$ is the difference of temperature between the plate and the fluid at a short distance from the plate The term h is known as the "coefficient of heat transfer" Its dimension is evidently $[FL^{-1}T^{-1}\Theta^{-1}]$ or $[MT^{-3}\Theta^{-1}]$

53 Equation of State of a Perfect Gas

The pressure that a gas exerts on the wall of a container is caused by impacts of the molecules on the wall Consequently, the pressure p of a gas depends primarily on the mass m of a molecule, the average velocity V of the molecules, and the number n of molecules per unit volume, i e,

$$p = f(m, V, n)$$

The only dimensionally homogeneous equation among these variables is

$$p = KmnV^2 \tag{a}$$

in which K is a dimensionless constant

According to the kinetic theory of gases, the absolute temperature θ is proportional to the mean kinetic energy of a single molecule, i e,

$$\theta = K_1 m V^2$$

in which K_1 is a constant Also, the mass density ρ is evidently equal to mn Consequently, Equation a may be written

$$p = \frac{K_2 \rho\theta}{m} \tag{b}$$

in which K_2 is a constant

The mass m of a molecule is proportional to the molecular weight M of the gas Consequently, Equation b yields

$$p = \frac{\rho R\theta}{M} \tag{c}$$

in which R is a constant that has the same value for all gases Equation c is known as the "equation of state" for gases

The molecular weight M is defined to be thirty-two times the ratio of the mass of a molecule of the given gas to the mass of a molecule of oxygen Since M is a ratio of masses, it is dimensionless Accordingly, Equation c shows that the gas constant R has the dimension of specific heat, i e $[R] = [L^2T^{-2}\Theta^{-1}]$

If Equation c is applied to a mixture of gases, such as air, an average molecular weight may be employed [21] The molecular weight of air is 28 8

In American engineering texts on thermodynamics, mass density is com-

[21] Max Planck *Theory of Heat* (Vol V of *Introduction to Theoretical Physics*) Ch I, Macmillan New York 1932

monly expressed in the unit "lb/ft³." Since, in gas dynamics, the distinction between mass and weight is important, writers on this subject frequently introduce the factor g in Equation c, to account for a change of the mass unit from the pound to the slug. However, gravity manifestly has nothing to do with the equation of state of a gas. The physical content of the equation of state is clearly exhibited by Equation c. Since dynamics plays a genuine role in modern applications of thermodynamics, simplicity and clarity will be enhanced if the units that are used are reconciled with Newton's law, $F = ma$. This implies that the pound should not be used simultaneously as a unit of force and a unit of mass. When the units are consistent with the equation, $F = ma$, the value of R does not depend on the units of force or mass, since R does not contain the dimensions of force or mass. The value of R is

$$R = 8.31 \times 10^7 \text{ cm}^2/\text{sec}^2 \text{ deg C} = 49{,}600 \text{ ft}^2/\text{sec}^2 \text{ deg F}$$

Equation c is inaccurate if the condition of liquefaction is approached. This is explained by the circumstance that the preceding analysis does not account for intermolecular forces nor for the finite diameters of the molecules. More accurate equations of state (e.g. van der Waals' equation) include terms that account for these variables.

54. Standard Dimensionless Products in the Theory of Heat

If the set of variables in Article 7 is augmented by temperature θ, the heat transfer coefficient h, the coefficient of thermal conductivity k, specific heat C, and the coefficient of thermal expansion β, the following complete set of dimensionless products is obtained:

Reynolds' Number $\quad \mathbf{R} = \dfrac{VL\rho}{\mu} = \dfrac{VL}{\nu}$

Pressure Coefficient $\quad \mathbf{P} = \dfrac{F}{\rho V^2 L^2} = \dfrac{p}{\rho V^2}\cdot \quad (p = \text{pressure})$

Froude's Number $\quad \mathbf{F} = \dfrac{V^2}{Lg}$

Mach's Number $\quad \mathbf{M} = \dfrac{V}{c}$

Weber's Number $\quad \mathbf{W} = \dfrac{\rho V^2 L}{\sigma}$

Grashof's Number $\quad \mathbf{G} = \dfrac{\beta\theta g L^3 \rho^2}{\mu^2}$

Nusselt's Number $\mathbf{N} = \dfrac{hL}{k}$

Prandtl's Number $\mathbf{Q} = \dfrac{C\mu}{k}$

No Name $\mathbf{K} = \dfrac{k\theta}{\rho V^3 L}$

The variable **K** is not a standard dimensionless product It is introduced in order that the above set of dimensionless products shall be complete

The dimensionless product $C\rho VL/k$ is known as "Peclet's number" This product is identical to **QR**

55 HEAT TRANSFER TO A FLOWING FLUID IN A PIPE

Consider the turbulent flow of a fluid in a smooth pipe of diameter D The mean velocity is V, and the mean temperature on a given cross section is θ The wall temperature is $\theta + \Delta\theta$ The mass density of the fluid is ρ, its kinematic coefficient of viscosity is ν, its thermal conductivity is k, and its specific heat is C The heat transmitted through the wall per unit area and per unit time is denoted by $h\Delta\theta$

Only differences of temperature affect the flow of heat Consequently, the heat transfer coefficient h does not depend on the temperautre θ

Near the wall of the pipe, there is a thin layer of fluid in which the flow is nearly laminar (laminar sublayer) Since the heat is conveyed through this layer primarily by conduction, the coefficient of thermal conductivity k is an important variable for determining the heat transfer coefficient h The thickness of the laminar sublayer is influenced by the kinematic viscosity ν, the diameter D and the mean velocity V These variables also affect the flow outside of the laminar sublayer

The steady flow of heat by conduction is not affected by specific heat However, in convection processes the specific heat is significant, since the amount of heat that is conveyed by a particle of fluid depends on its heat capacity A deeper insight into this relationship is obtained by studying the derivation of the differential equation for heat transfer in a laminar velocity field Since the turbulent flow of a fluid with nonuniform temperature entails a large amount of heat transfer by convection, the heat capacity of the fluid enters into analyses of this type of phenomenon The relevant variable is the heat capacity per unit volume This is represented by $C\rho$, in which C is the heat capacity per unit mass (specific heat)

In view of the preceding remarks, the heat transfer coefficient h is determined by the variables V, D, ν, k $C\rho$, and $\Delta\theta$ The rank of the dimensional matrix is four Consequently, there are three dimensionless products in a complete set. A complete set of dimensionless products consists of the Nusselt number $\mathbf{N} = hD/k$, the Reynolds number $\mathbf{R} = VD/\nu$, and the Prandtl number $\mathbf{Q} = C\rho\nu/k$. Since these products do not contain the temperature difference $\Delta\theta$, it is impossible to form a dimensionless product that contains $\Delta\theta$. Therefore, h does not depend on $\Delta\theta$.

Buckingham's theorem now yields

$$\mathbf{N} = f(\mathbf{R}, \mathbf{Q}) \quad \text{or} \quad h = \frac{k}{D} f(\mathbf{R}, \mathbf{Q})$$

Experiments[22] indicate that the function $f(\mathbf{R}, \mathbf{Q})$ is approximated by $0.023\mathbf{R}^{0.8}\mathbf{Q}^{0.4}$.

56. Condensation in a Vertical Pipe

Consider vapor at the saturation temperature θ passing through a smooth vertical pipe whose wall temperature is $\theta - \Delta\theta$. The condensate forms a film on the wall that is an insulating layer. Consequently, the rate of condensation is influenced by the coefficient of thermal conductivity k of the condensate. The rate of condensation is determined directly by the average heat-transfer coefficient h, since the heat that is extracted from the vapor per unit time is $hA\Delta\theta$, in which A is the area of the wall of the pipe.

The main geometrical variable in the problem is the thickness of the film of condensate. This depends on the rate of condensation and the nature of the flow of condensate. The rate of condensation depends on the latent heat of vaporization of the fluid. Since the volume, rather than the mass, of condensate is significant, the latent heat should be expressed as "heat of vaporization per unit volume." This is represented by $\rho\lambda$, in which λ is the latent heat of vaporization per unit mass and ρ is the mass density of the condensate.

The facility with which the film of condensate flows from the wall is determined mainly by its viscosity μ and its specific weight ρg. Also, since the thickness of the film is not constant along the pipe, the length L of the pipe affects the coefficient of heat transfer. The diameter of the pipe does not affect the thickness of the film (and consequently does not affect h) if it is large compared to the thickness of the film. The velocity of the vapor in the pipe influences the thickness of the film to some extent, but this effect is small if the velocity is not large. If the interaction between the flow of vapor and the flow of condensate is neglected, the density of the vapor is irrelevant.

In view of the preceding discussion, there is a relationship of the form,

$$f(h, \Delta\theta, L, \rho\lambda, k, \rho g, \mu) = 0 \tag{a}$$

[22] W. H. McAdams, Review and Summary of Developments in Heat Transfer by Conduction and Convection, *Trans. Am. Inst. Chem. Engrs.*, Vol. 36, no. 1, 1940.

The dimensional matrix is

	1	2	3	4	5	6	7
	h	$\Delta\theta$	L	$\rho\lambda$	k	ρg	μ
M	1	0	0	1	1	1	1
L	0	0	1	−1	1	−2	−1
T	−3	0	0	−2	−3	−2	−1
Θ	−1	1	0	0	−1	0	0

The rank of the matrix is four Consequently, there are three dimensionless products in a complete set The equations corresponding to the dimensional matrix are

$$k_1 + k_4 + k_5 + k_6 + k_7 = 0$$

$$k_3 - k_4 + k_5 - 2k_6 - k_7 = 0$$

$$-3k_1 - 2k_4 - 3k_5 - 2k_6 - k_7 = 0$$

$$-k_1 + k_2 - k_5 = 0$$

The matrix of solutions is

	1	2	3	4	5	6	7
	h	$\Delta\theta$	L	$\rho\lambda$	k	ρg	μ
π_1	1	0	0	1	−1	−1	0
π_2	0	1	0	−4	1	2	1
π_3	0	0	1	−1	0	1	0

Accordingly, a complete set of dimensionless products is

$$\pi_1 = \frac{h\lambda}{kg}, \qquad \pi_2 = \frac{k\mu g^2\Delta\theta}{\rho^2\lambda^4}, \qquad \pi_3 = \frac{gL}{\lambda}$$

By Buckingham's theorem, Equation a must now reduce to the form,

$$f(\pi_1, \pi_2, \pi_3) = 0 \quad \text{or} \quad \pi_1 = f(\pi_2, \pi_3)$$

Hence,

$$\frac{h\lambda}{kg} = f\left(\frac{k\mu g^2\Delta\theta}{\rho^2\lambda^4}, \frac{gL}{\lambda}\right) \tag{b}$$

By virtue of this equation, the phenomenon may be completely described by a chart in which the ordinate is $h\lambda/kg$, the abscissa is $k\mu g^2\Delta\theta/(\rho^2\lambda^4)$, and the parameter of the curves is gL/λ In the experimental construction of the chart, π_2 may be varied by varying the temperature difference $\Delta\theta$, and π_3 may be varied by varying the length L Note the great simplification that results by reducing the number of independent variables from six to two

W. Nusselt[23] analyzed the phenomenon, using the assumption that the flow of the film of condensate is laminar. He arrived at the formula:

$$h = 0.943 \sqrt[4]{\frac{g\rho^2\lambda k^3}{L\mu\Delta\theta}}$$

In terms of the dimensionless products that have been found, this equation may be expressed

$$\pi_1 = \frac{0.943}{\sqrt[4]{\pi_2\pi_3}}$$

This is a special form of Equation b.

In some cases, condensate is observed to form in drops rather than in a film. To take this phenomenon into account, it is necessary to introduce the surface tension in the analysis. Also, the roughness and the cleanness of the wall affect drop-type condensation.

57. Transient Heat Transmission to a Body in a Fluid

Suppose that a solid conducting body, with initial temperature θ_0, is immersed in a large bath of fluid at temperature θ_f and that the fluid is stirred, so that its temperature remains practically uniform and constant. Then the temperature θ at any specified point in the body varies with time, and it approaches the temperature θ_f asymptotically.

Only differences of temperature are significant in heat-transfer processes. In the present case, the differences $\theta - \theta_0$ and $\theta_f - \theta_0$ determine the remaining difference $\theta_f - \theta$. The response of the temperature to an influx of heat depends on the heat capacity of the body per unit volume, $C\rho$.

In general, there is a discontinuity of temperature $\Delta\theta$ at a boundary between two different substances. The time rate of heat flow through a unit area of the boundary is $h\Delta\theta$, in which the heat-transfer coefficient h is a constant for the boundary.

In view of these remarks, the temperature θ at a specified point in the body at the time t is determined by an equation of the form,

$$\theta - \theta_0 = f(k, h, C\rho, L, t, \theta_f - \theta_0)$$

in which k is the coefficient of thermal conductivity of the body and L is a length that designates the size of the body. Dimensional analysis of this equation yields

$$\frac{\theta - \theta_0}{\theta_f - \theta_0} = f\left(\frac{kt}{C\rho L^2}, \frac{hL}{k}\right) \qquad \text{(a)}$$

This equation was derived by Gurney and Lurie[24] in 1923.

[23] W. Nusselt, *Z. Ver. deut. Ing.*, Vol. 60, pp. 541 and 569, 1916.
[24] *Ind. Eng. Chem.*, Vol. 15, p. 1173, 1923.

Equation a has a property that enables us to deduce the precise manner in which the time variable enters the equation For, irrespective of the value of θ_0, the variable t represents the time interval in which the temperature changes from θ_0 to θ Consequently, Equation a may be written

$$\frac{\Delta\theta}{\theta_f - \theta} = f\left(\frac{k\Delta t}{C\rho L^2}, \frac{hL}{k}\right) \tag{b}$$

wherein Δt is the time interval in which the temperature changes from θ to $\theta + \Delta\theta$ Letting Δt be an infinitesimal, and expanding Equation b by means of MacLaurin's series, we get

$$\frac{\Delta\theta}{\theta_f - \theta} = f\left(0, \frac{hL}{k}\right) + \frac{k\Delta t}{C\rho L^2} f'\left(0, \frac{hL}{k}\right) +$$

Since $\Delta\theta = 0$ if $\Delta t = 0$ the first term in this series is zero It follows that

$$\frac{d\theta}{\theta_f - \theta} = \frac{Ak\,dt}{C\rho L^2}$$

in which A is a function of hL/k Integration of this equation yields

$$\ln(\theta - \theta_f) = -\frac{Akt}{C\rho L^2} + \text{constant}$$

The constant must be chosen to render the equation dimensionally homogeneous and to satisfy the initial condition $\theta = \theta_0$ for $t = 0$ These conditions yield

$$\frac{Akt}{C\rho L^2} = \ln\frac{\theta_f - \theta_0}{\theta_f - \theta} \tag{c}$$

This equation determines the time t in which the temperature changes from θ_0 to θ Inversion of Equation c yields

$$\frac{\theta - \theta_0}{\theta_f - \theta_0} = 1 - e^{-Akt/C\rho L^2} \tag{d}$$

Thus, Equation a has been reduced to a more special form Equation d is an invaluable aid for understanding phenomena such as heat treatment of metals and chemical changes that occur during transient heating processes

58 Natural Convection

Consider a horizontal pipe with diameter d that is concentric with a larger pipe with diameter D The two pipes are maintained at the respective temperatures θ_0 and θ_1 The annular space between the pipes is filled with a liquid. Steady convection currents are generated by the thermal expansion of the liquid, but there is no axial flow. The rate of heat transfer from one pipe to the other may be expressed in the form, $hA(\theta_0 - \theta_1)$, in which A is the lateral area of the smaller pipe.

The motion of any particle of the fluid is governed by Newton's law, $a = F/m$. The force F is partly viscous friction and partly weight. The viscous friction is proportional to the coefficient of viscosity μ, and the mass m is proportional to the mass density ρ. Consequently, the ratio F/m contains the ratio $\nu = \mu/\rho$. Also, the ratio F/m contains the ratio $g = w/\rho$. Accordingly, the kinematic viscosity ν and the acceleration of gravity g are the significant dynamical variables in the problem. However, the kinematic viscosity ν varies appreciably with temperature. If the temperature difference $\theta_0 - \theta_1$ is not too great, the kinematic viscosity ν may be approximated in the range θ_0 to θ_1 by the linear equation, $\nu = \nu_0 + \nu_1(\theta - \theta_0)$, in which ν_0 and ν_1 are characteristic constants of the fluid.

The pertinent thermal variables are the temperature difference $\theta_0 - \theta_1$ (hereafter denoted by $\Delta\theta$), the heat capacity per unit volume $C\rho$, the coefficient of thermal conductivity k, and the coefficient of thermal expansion β. It follows that

$$h = f\,(d, D, \Delta\theta, C\rho, k, \beta, \nu_0, \nu_1, g)$$

There are ten variables in this equation, and the rank of the dimensional matrix is four. Consequently, there are six dimensionless products in a complete set. The following complete set of dimensionless products may be found by inspection:

$$\frac{hd}{k}, \quad \frac{\beta\Delta\theta g d^3}{\nu_0^2}, \quad \frac{C\rho\nu_0}{k}, \quad \frac{d}{D}, \quad \frac{\nu_0}{\nu_1\Delta\theta}, \quad \beta\Delta\theta$$

The first three of these products are Nusselt's number, Grashof's number and Prandtl's number.

Since the convection currents result from the joint action of gravity and thermal expansion, the product $\beta\Delta\theta$ presumably has no effect. This conjecture is confirmed by experiment. Accordingly, the following equation is obtained:

$$\frac{hd}{k} = f\left(\mathbf{G}, \mathbf{Q}, \frac{d}{D}, \frac{\nu_0}{\nu_1\Delta\theta}\right)$$

in which the notations **G** and **Q** are used for Grashof's number and Prandtl's number. Thus, the number of independent variables in the problem is reduced from nine to four.

Problems

1 Prove that (R, P, F, M, W, G, N, Q, K) is a complete set of dimensionless products

2 Make a dimensional analysis of the equation,

$$\theta = f(C, k, \rho, \mu, V, L)$$

3 Make a dimensional analysis of the equation,

$$h = f(\beta, \theta, g, \rho, C, k, L, t)$$

4 Carnot showed that the efficiency η of an ideal engine depends only on the absolute temperature θ_1 of the heat source and the absolute temperature θ_2 of the heat receiver What is the general form of the relationship? Given that the relationship is linear, and that $\eta = 1$ for $\theta_2 = 0$ and $\eta = 0$ for $\theta_1 = \theta_2$, derive the precise formula for the efficiency of an ideal engine

5 Show that, if the diameter d of a molecule affects the equation of state of a gas, then the gas constant R is a function of $\rho d^3/m$, in which m is the mass of a molecule and ρ is the mass density of the gas

6 Make a dimensional analysis of the equation

$$E = f(C\rho, k, \nu, w, \beta)$$

in which E denotes energy and w denotes specific weight

7 Let the diameter D of a tube be the characteristic length for a class of geometrically similar water tube boilers Assume that the average heat transfer coefficient h that defines the heat transfer from the gases to the tubes is a function of V, D ν, k ρ, and C, in which V is the velocity of gases in the flue Perform a dimensional analysis of the problem.

8 A cylindrical body of length L and diameter D is immersed in a stationary fluid The difference between the temperature of the body and the temperature of the fluid at a short distance from the body has a constant value $\Delta\theta$ Heat is transferred from the body to the fluid by direct conduction, and by convection currents that are set up by the thermal expansion of the fluid List the variables that determine the average heat transfer coefficient h Make a dimensional analysis of the problem

9 A chilled metal ball is dropped into a large tank of warm liquid The liquid is stirred, so that its temperature remains practically uniform and constant. List the variables that determine the thermal strain ϵ that exists at the center of the ball, t seconds after the ball is dropped into the liquid Make a dimensional analysis of the problem

10 A body at 10°C is dropped into a large bath of liquid which is maintained at a uniform temperature of 100°C In 5 min, the temperature at a point in the body rises to 40°C In how many more minutes will the temperature rise from 40°C to 98°C?

CHAPTER 9

Dimensional Treatment of Problems of Electromagnetic Theory

59. Introduction

In this chapter, it must be presupposed that the reader is familiar with the elements of electromagnetic theory.

Several different systems of units have evolved in electromagnetic theory, and the dimensions of electrical and magnetic entities depend on which system is used. The rationalized Giorgi system[25] is now accepted by electrical engineers, and it is gaining favor among physicists. Only this system is considered in the following.[26]

Maxwell's electromagnetic theory is the foundation of much of the theory that is used in electrical engineering. In the words of Heinrich Hertz, "Maxwell's theory is best defined as Maxwell's equations." In the rationalized Giorgi system, these equations are expressed as follows:

$$\text{curl } E + \frac{\partial B}{\partial t} = 0 \tag{29}$$

$$\text{curl } H - \frac{\partial D}{\partial t} = J \tag{30}$$

The vector functions E, H, J, D, B are known as "electric field intensity," "magnetic field intensity," "electric current density," "electric displacement" (or electric induction), and "magnetic induction," respectively. The mathematical consequences of Maxwell's equations may be deduced without inquiring into the physical meanings of the vector fields E, H, J, D, and B. However, the importance of Maxwell's theory in engineering and in physics naturally rests on the circumstance that these vectors have

[25] An explanation of the Giorgi system has been given by J. Stratton, *Electromagnetic Theory*, Article 1.8, McGraw-Hill, New York, 1941.

[26] For a discussion of dimensions in the Gaussian system of measurement, see Max Planck, *Theory of Electricity and Magnetism* (Vol. III of *Introduction to Theoretical Physics*), Macmillan, New York, 1932.

been identified with measurable quantities The manner in which the correlation is achieved is described in elementary expositions of electricity and magnetism

In most isotropic, nonmagnetic materials, the five vectors E, H, J, D, B are linearly related, as follows

$$D = \epsilon E, \qquad B = \mu H, \qquad J = \kappa E \tag{31}$$

The scalars ϵ, μ, and κ are characteristic constants of the material, respectively, called "electric inductive capacity," "magnetic inductive capacity," and "specific electric conductivity"

The vector J is conceived to represent flow of electricity More precisely, the quantity of electricity that flows through an elemental area dA in an infinitesimal time interval dt is $J_n\, dA\, dt$, in which J_n is the component of J on the normal to the element dA. Since electricity is conserved, this interpretation leads directly to the equation,

$$\frac{dq}{dt} = -\oint_A J_n\, dA \tag{32}$$

in which J_n is the component of J on the outward directed normal to any closed surface A, and q is the total electric charge within the region that is enclosed by the surface A

If charged conductors in an electrostatical field are given slight displacements, the principle of conservation of energy requires that the work that is supplied to move the conductors shall equal the increase of electric energy in the field Deductions based on this principle lead to Coulomb's law,

$$F = \frac{qq'}{4\pi\epsilon r^2} \tag{33}$$

Here, F denotes the force that acts between two concentrated charges, q and q', that are separated by a distance r

60 THE UNIT OF ELECTRIC CHARGE

In order to link the foregoing equations with experimental facts, we must choose a unit for one electrical quantity The quantity that is selected for this purpose is arbitrary Analogously, in mechanics, two systems of dimensions—the force system and the mass system—have evolved, because units may be arbitrarily assigned to either force or mass The current trend in electrical engineering is to regard electric charge as a fundamental quantity Accordingly, a new dimensional symbol [Q] is employed for electric charge An equation will be said to be dimensionally homogeneous if and only if, it remains unchanged when the units of mass, length, time, temperature and electric charge are changed in any way

The standard unit of electric charge is the *coulomb*. The coulomb may be defined by the condition that two unit charges in a vacuum, when separated by a distance of one meter, exert a force of 9×10^9 newtons upon each other. (The definition of a newton is given in Article 2.) The coulomb is evidently an enormous quantity of electricity. Nevertheless, the ampere, a practical unit of current, is defined to be a flow of electricity of magnitude one coulomb per second.

In the Gaussian system of measurement, the unit of charge is defined by the condition that two unit charges in a vacuum, when separated by a distance of one centimeter, exert a force of one dyne upon each other. Accordingly, in the Gaussian system, Coulomb's law (Equation 33) becomes

$$F = \frac{qq'}{r^2}$$

This equation shows that the dimension of electric charge, in the Gaussian system, is $[F^{1/2}L]$ or $[M^{1/2}L^{3/2}T^{-1}]$. The reasoning that has led to this conclusion is analogous to the reasoning that shows that the dimension of mass in the astronomical system is $[L^3T^{-2}]$ (see Example 3).

61. Numerical Values of ϵ_0 and μ_0

If the field vectors E and H are functions of a single space coordinate x, and if the material does not conduct electricity (i.e., $\kappa = 0$), Equations 29, 30 and 31 lead directly to the wave equation for the field components; namely:

$$\frac{\partial^2 E_y}{\partial t^2} = \frac{1}{\epsilon\mu}\frac{\partial^2 E_y}{\partial x^2}, \quad \text{etc.}$$

Consequently, the speed of a plane electromagnetic wave in a nonconductor is $1/\sqrt{\epsilon\mu}$. This is the speed of light in the medium. It follows that

$$1/\sqrt{\epsilon_0\mu_0} = 3 \times 10^8 \text{ m/sec} \tag{34}$$

in which ϵ_0 and μ_0 are the values of ϵ and μ for empty space.

It is apparent from Equation 33 that the dimension of ϵ is

$$[\epsilon] = [M^{-1}L^{-3}T^2Q^2]$$

Hence, by Equation 34,

$$[\mu] = [MLQ^{-2}]$$

Adopting the coulomb as the unit of charge, we obtain, from Equation 33,

$$\epsilon_0 = (36\pi \times 10^9)^{-1}\left[\frac{\text{sec}^2\ \text{coul}^2}{\text{kg m}^3}\right]$$

It follows, from Equation 34,

$$\mu_0 = 4\pi \times 10^{-7} \left[\frac{\text{kg m}}{\text{coul}^2}\right]$$

Thus, the inductive capacities of empty space are determined

62 Dimensions of Electrical Entities

The dimensions of ϵ and μ have been derived in the preceding article

By Equation 32, the dimension of electric current density is

$$[J] = [L^{-2}T^{-1}Q]$$

The integral of the current density J over the cross section of a wire is the total current that is flowing in the wire Consequently, the dimension of electric current I is

$$[I] = [T^{-1}Q]$$

Equation 30 now shows that the dimension of electrical displacement D is

$$[D] = [L^{-2}Q]$$

Then, in view of Equation 31 the dimension of electric field intensity E is

$$[E] = [MLT^{-2}Q^{-1}] = [FQ^{-1}]$$

In view of Equation 29, any stationary electric field is derivable from a potential function, i e ,

$$E = -\text{grad}\,\phi$$

The scalar function ϕ is called "electric potential" Evidently, the dimension of ϕ is

$$[\phi] = [ML^2T^{-2}Q^{-1}]$$

The line integral of the vector E is called "electromotive force" This has the same dimension as ϕ

The capacitance C of a capacitor is the rate of increase of charge with respect to potential i e , $C = dq/d\phi$ Consequently, the dimension of electric capacitance is

$$[C] = [M^{-1}L^{-2}T^2Q^2]$$

Since the dimensions of electromotive force and electric current have been determined, the dimension of electrical resistance R is determined by Ohm's law Accordingly,

$$[R] = [ML^2T^{-1}Q^{-2}]$$

The work that a stationary electric field performs on a concentrated charge q as it moves from point 1 to point 2 is

$$q\int_1^2 E \cdot ds = q(\phi_1 - \phi_2)$$

in which ϕ_1 and ϕ_2 are the potentials at the respective points. Consequently, if the potential drop in a linear conductor is ϕ, the work that the electrical forces perform per unit time is ϕI, in which I is the current in the conductor. Accordingly, ϕI is the electric power that is dissipated into heat. Note that this product has the dimension of power.

63. Dimensions of Magnetic Entities

In a stationary field, the line integral of the magnetic field intensity around any closed curve equals the electric current that flows through the loop. This follows from Equation 30, with the aid of Stokes's theorem. In symbols,

$$\oint H \cdot ds = I$$

in which the left-hand term represents the line integral of the magnetic field intensity H about a given closed curve, and I represents the current that links with the curve. Accordingly, the dimension of magnetic field intensity is

$$[H] = [L^{-1}T^{-1}Q]$$

It has been shown in Article 61 that the dimension of the constant μ is

$$[\mu] = [MLQ^{-2}]$$

Consequently, the dimension of magnetic induction, $B = \mu H$, is

$$[B] = [MT^{-1}Q^{-1}]$$

The surface integral of the normal component of the vector B over any surface that caps a closed curve (circuit) is called the "flux of magnetic induction through the surface" or the "flux of magnetic induction that links with the circuit." Consequently, the dimension of flux of magnetic induction is $[ML^2T^{-1}Q^{-1}]$. In view of Equation 29, the time rate of change of flux of magnetic induction is the induced electromotive force in the circuit. It may be directly verified that this relationship is consistent with the dimensions of magnetic induction and electromotive force.

A current I flowing in a circuit causes a proportionate flux of magnetic induction that links with the circuit. If the current is changed, the flux of induction through the circuit is changed, and, in view of the preceding remarks, an electromotive force is induced in the circuit. Since the rate of change of flux of induction is proportional to the rate of change of current, the induced electromotive force is $\mathfrak{L}\ dI/dt$, in which $\mathfrak{L}$ is a constant. The factor $\mathfrak{L}$ is called the "coefficient of self-inductance" of the circuit. Evidently, the dimension of $\mathfrak{L}$ is $[ML^2Q^{-2}]$.

A current I flowing in a circuit causes a flux of magnetic induction through an adjacent circuit Consequently, if the current is changed, an electromotive force is induced in the adjacent circuit The induced electromotive force may be expressed in the form $\mathfrak{M}\, dI/dt$ The factor $\mathfrak{M}$ is called the 'coefficient of mutual inductance" of the circuits The coefficient of mutual inductance has the same dimension as the coefficient of self inductance

EXAMPLE 16 COEFFICIENT OF INDUCTANCE

In a class of geometrically similar circuits, a circuit is specified by a single length L The coefficient of self inductance is then determined by the length L and by the magnetic inductive capacity μ of the medium in which the circuit lies Hence,

$$f(\mathfrak{L}, L, \mu) = 0$$

The dimensional matrix is

	$\mathfrak{L}$	L	μ
M	1	0	1
L	2	1	1
T	0	0	0
Q	−2	0	−2

The rank of this matrix is two Consequently, there is only one dimensionless product in a complete set By inspection the product is found to be $\mathfrak{L}/\mu L$ Consequently, the only dimensionally homogeneous relationship among the variables is

$$\mathfrak{L} = K\mu L$$

in which K is a dimensionless factor that depends on the shape of the circuit Accordingly, the self inductance of a circuit is proportional to the size of the circuit

64 STANDARD ELECTRICAL UNITS

In the practical system of electrical measurement, the MKS mass system (Article 2) is employed The unit of force is then the newton The standard unit of electric charge is the coulomb Then in view of the dimensions that have been derived, the practical unit of electric field intensity is

$$1\left[\frac{\text{newton}}{\text{coulomb}}\right] = 1\left[\frac{\text{kg m}}{\text{sec}^2\ \text{coul}}\right]$$

The unit of electric potential is then

$$1\left[\frac{\text{newt m}}{\text{coul}}\right] = 1\left[\frac{\text{kg m}^2}{\text{sec}^2\ \text{coul}}\right] = 1 \text{ volt}$$

The unit of electric capacitance is

$$1\left[\frac{\text{coul}}{\text{volt}}\right] = 1\left[\frac{\text{coul}^2\ \text{sec}^2}{\text{kg m}^2}\right] = 1 \text{ farad}$$

Since the farad is an enormously large unit, capacitances are commonly expressed in microfarads (i.e. millionths of a farad).

The practical unit of electric current is

$$1\left[\frac{\text{coul}}{\text{sec}}\right] = 1 \text{ ampere}$$

The unit of electrical resistance is then

$$1\left[\frac{\text{volt}}{\text{ampere}}\right] = 1\left[\frac{\text{kg m}^2}{\text{sec coul}^2}\right] = 1 \text{ ohm}$$

The unit of electric power is

$$1 \text{ ampere volt} = 1\left[\frac{\text{kg m}^2}{\text{sec}^3}\right] = 1\left[\frac{\text{newton meter}}{\text{sec}}\right] = 1\left[\frac{\text{joule}}{\text{sec}}\right] = 1 \text{ watt}$$

The unit of magnetic field intensity H is

$$1\left[\frac{\text{coul}}{\text{m sec}}\right]$$

and the unit of magnetic inductive capacity μ is

$$1\left[\frac{\text{kg m}}{\text{coul}^2}\right]$$

Consequently, the unit of magnetic induction B is

$$1\left[\frac{\text{kg}}{\text{sec coul}}\right]$$

and the unit of flux of magnetic induction is

$$1\left[\frac{\text{kg m}^2}{\text{sec coul}}\right] = 1 \text{ weber}$$

Finally, the unit of the coefficient of self-inductance or mutual inductance is

$$1\left[\frac{\text{kg m}^2}{\text{coul}^3}\right] = 1 \text{ henry}$$

The preceding results indicate that some electrical units are simple combinations of other units For example,

$$1 \text{ kg} = 1 \text{ watt sec}^3/\text{m}^2$$

$$1 \text{ watt} = 1 \text{ amp volt}$$

$$1 \text{ farad} = 1 \text{ amp sec/volt}$$

$$1 \text{ ohm} = 1 \text{ volt/amp}$$

$$1 \text{ weber} = 1 \text{ volt sec}$$

$$1 \text{ henry} = 1 \text{ volt sec/amp} = 1 \text{ weber/amp}$$

$$1 \text{ joule} = 1 \text{ watt sec}$$

In electrical engineering the concept of mass is not used extensively Consequently, it is customary to adopt electric potential or some other electrical quantity as a fundamental dimension, rather than mass Also, since electrostatic fields are comparatively unimportant in electrical engineering, it is convenient, in practice to employ electric current as a fundamental dimension, rather than electric charge Since the product of current and voltage is electric power, the dimension of mass is then $[M] = [L^{-2}T^3I\Phi]$, in which $[I]$ denotes electric current and $[\Phi]$ denotes electric potential

Table 3 gives dimensions of electrical quantities in the $MLTQ$ system and in the $LTI\Phi$ system

TABLE 3

DIMENSIONS AND UNITS OF ELECTRICAL AND MAGNETIC ENTITIES

	$MLTQ$ System	$LTI\Phi$ System	Name of Unit
Mass	$[M]$	$[L^{-2}T^3I\Phi]$	kg
Electric Charge	$[Q]$	$[TI]$	coulomb
Electric Inductive Capacity ϵ	$[M^{-1}L^{-3}T^2Q^2]$	$[L^{-1}TI\Phi^{-1}]$	farad/m
Magnetic Inductive Capacity μ	$[MLQ^{-2}]$	$[L^{-1}TI^{-1}\Phi]$	ohm sec/m
Electric Current Density J	$[L^{-2}T^{-1}Q]$	$[L^{-2}I]$	amp/m²
Electric Current	$[T^{-1}Q]$	$[I]$	amp
Electric Displacement D	$[L^{-2}Q]$	$[L^{-2}TI]$	amp sec/m²
Electric Field Intensity E	$[MLT^{-2}Q^{-1}]$	$[L^{-1}\Phi]$	volt/m
Electric Potential	$[ML^2T^{-2}Q^{-1}]$	$[\Phi]$	volt
Electric Capacitance	$[M^{-1}L^{-2}T^2Q^2]$	$[TI\Phi^{-1}]$	farad
Electric Resistance	$[ML^2T^{-1}Q^{-2}]$	$[I^{-1}\Phi]$	ohm
Magnetic Field Intensity H	$[L^{-1}T^{-1}Q]$	$[L^{-1}I]$	amp/m
Magnetic Induction B	$[MT^{-1}Q^{-1}]$	$[L^{-2}T\Phi]$	weber/m²
Flux of Magnetic Induction	$[ML^2T^{-1}Q^{-1}]$	$[T\Phi]$	weber
Coefficient of Inductance $\mathfrak{L}$, $\mathfrak{M}$	$[ML^2Q^{-2}]$	$[TI^{-1}\Phi]$	henry
Electric Energy	$[ML^2T^{-2}]$	$[TI\Phi]$	joule
Electric Power	$[ML^2T^{-3}]$	$[I\Phi]$	watt

65. FERROMAGNETISM

In iron and steel, the vectors B and H do not necessarily have the same direction, nor is the ratio of the magnitudes of these vectors constant. In fact, the phenomenon of hysteresis shows that B is not uniquely determined by H, unless the field increases monotonically. The relationship between B and H in a ferromagnetic material is thus similar to the relationship between stress and strain in an inelastic material. For a monotonically increasing field, the relationship between the magnitudes of the vectors B and H may be represented by a graph that is analogous to the stress-strain curve of a ductile metal. The general form of this curve for a ferro-

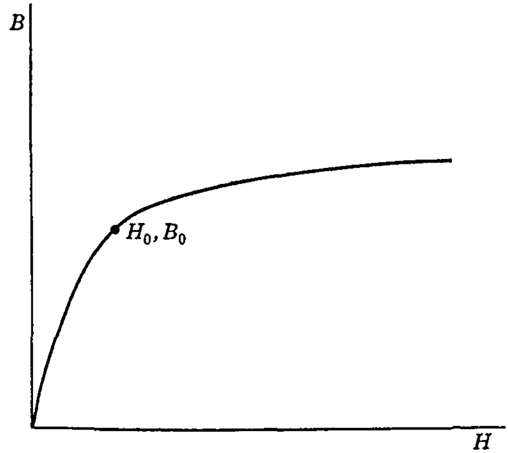

FIG. 14. Magnetization Curve for a Ferromagnetic Material

magnetic substance is shown in Figure 14. Letting (H_0, B_0) be a particular point on the graph (e.g. the knee of the curve), we may plot the relationship in a dimensionless form by employing the abscissa H/H_0 and the ordinate B/B_0. In applying dimensional analysis to problems involving ferromagnetism, we must restrict attention to a class of substances whose dimensionless (B, H) curves are identical. In this class, a material is characterized by the constants H_0 and B_0. Analogously, in problems concerned with inelastic properties of materials, attention is directed to a class of materials with a common dimensionless stress-strain curve. In this class, a material is characterized by its elastic constants.

EXAMPLE 17. THERMISTORS*

A thermistor is an electric conductor whose resistance decreases markedly as the temperature rises. A common type of thermistor is a bead or a disk

* Examples 17 and 18 were contributed by Mr. Knute J. Takle of the Naval Electronics Laboratory in San Diego, California.

made by sintering a number of metallic oxides into a compact mass Thermistors have recently had a number of applications to electric circuits requiring variable resistors It has been found that the resistance R of a thermistor is related to the absolute temperature θ by an equation of the following type

$$R = \alpha e^{\beta/\theta} \tag{a}$$

The terms α and β are characteristic constants of the thermistor, and e is the base of natural logarithms

The potential drop ϕ across a thermistor is determined by Ohm's law, $\phi = IR$ However, the resistance R depends on the equilibrium temperature that prevails when the current is flowing This, in turn, is determined by the rate of heat transfer to the surrounding medium The heat that is transferred to the surrounding medium per second (watts) is denoted by $h\Delta\theta$, in which $\Delta\theta$ is the difference between the temperature of the thermistor and the ambient temperature The heat transfer coefficient h is practically a constant for a given thermistor, if heat is transferred primarily by conduction and convection If the temperature is so high that a large amount of heat is radiated from the thermistor, another factor enters the problem

It may now be concluded that the potential drop ϕ across a thermistor is determined by the current I, the constants α and β, the ambient temperature θ_0, and the heat transfer coefficient h Since the resistance R_0 at the ambient temperature is determined by the equation $R_0 = \alpha e^{\beta/\theta_0}$, the constant α may be replaced by R_0

The relationship among the variables is indicated by the following equation

$$\phi = f(I, \theta_0, R_0, \beta, h) \tag{b}$$

In the $LTI\Phi$ system of dimensions, the dimensional matrix is

	1	2	3	4	5	6
	ϕ	I	θ_0	R_0	β	h
I	0	1	0	−1	0	1
Φ	1	0	0	1	0	1
Θ	0	0	1	0	1	−1

The rank of this matrix is 3 The method described in Chapter 3 leads to the following complete set of dimensionless products

$$\pi_1 = \frac{\phi}{\sqrt{R_0\beta h}}, \qquad \pi_2 = I\sqrt{\frac{R_0}{\beta h}}, \qquad \pi_3 = \frac{\theta_0}{\beta}$$

Buckingham's theorem now yields $\pi_1 = f(\pi_2, \pi_3)$

Bollman and Kreer[27] have employed a different set of dimensionless products that may be expressed in terms of the preceding set as follows:

$$\pi'_1 = \frac{\pi_1}{\pi_3}, \qquad \pi'_2 = \frac{\pi_2}{\pi_3}, \qquad \pi'_3 = \pi_3$$

Accordingly, they have obtained the equation,

$$\frac{\phi}{\theta_0}\sqrt{\frac{\beta}{R_0 h}} = f\left(\frac{I}{\theta_0}\sqrt{\frac{\beta R_0}{h}}, \frac{\theta_0}{\beta}\right) \tag{c}$$

By an analytical treatment of the problem, Bollman and Kreer[27] have derived the precise form of Equation c, and they have presented the result

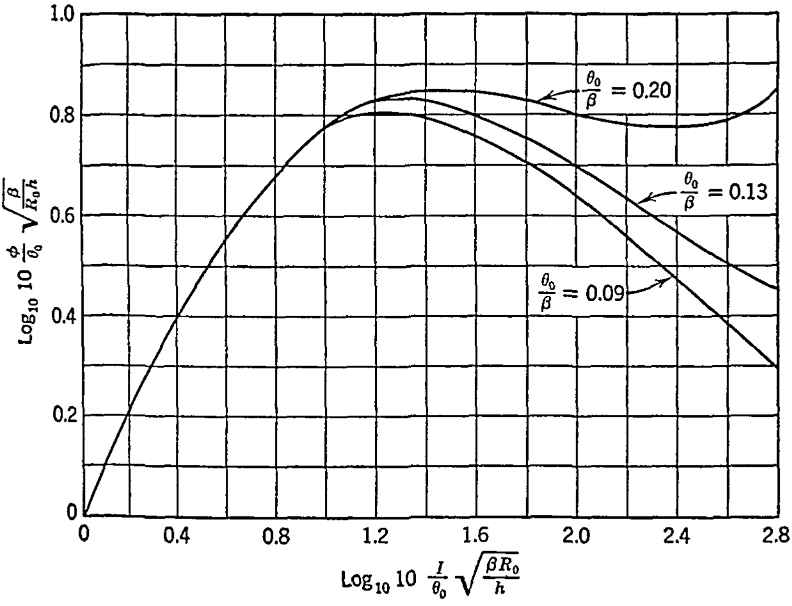

FIG. 15. Characteristics of Thermistors

By courtesy of the Institute of Radio Engineers (Reference 27).

by a chart (Fig. 15). This chart is valid for any thermistor whose resistance is given by Equation a and whose heat-transfer coefficient h is constant.

EXAMPLE 18. PIEZOELECTRIC RECEIVERS*

Certain crystals, known as "piezoelectric crystals" become electrified when they are strained. Conversely, they experience strains when they

[27] J. H. Bollman and J. G. Kreer, *Proc. IRE*, Vol. 38, no. 1, p. 20, Jan. 1950.

* Examples 17 and 18 were contributed by Knute J. Takle of the Naval Electonics Laboratory in San Diego, California.

are electrified In the absence of boundary stresses and thermal strains, the relationship between strain and electric field intensity in a piezoelectric crystal is linear Eighteen characteristic constants of the crystal (piezoelectric constants) enter this relationship Ratios of these constants yield seventeen independent dimensionless products These products may be disregarded if a specific type of crystal at a specific temperature is considered Then a single piezoelectric constant d adequately represents the piezoelectric properties of the material Since the product of the constant d with the electric field intensity E is a strain, the dimension of d is $[M^{-1}L^{-1}T^{2}Q]$

For a crystal the relationship between electric field intensity E and the electric displacement D involves nine characteristic coefficients, but again a single representative coefficient fulfills the needs of dimensional analysis Likewise, the mechanical properties of a crystal are adequately represented by the mass density ρ and a single elastic constant λ (λ has the dimension of pressure)

Piezoelectric crystals are used in some receivers or ' pick ups" that record mechanical vibrations The crystals in this type of instrument are usually cut to rectangular forms and they are arranged in stacks, with metal foil conductors between adjacent crystals The orientations of the planes of the cuts with respect to the crystallographic axes affect the behavior of the instrument The metal foil conductors are connected to wires that lead to a galvanometer The stacks of crystals are attached to one or more iron bars that do not vibrate appreciably The whole instrument may be enclosed in a container of oil that has a diaphragm or sonic window" through which the external vibrations enter This type of receiver is used for detecting underwater vibrations caused by submarines and other disturbances in the ocean

Suppose that a source of sound at a distance L from the receiver emanates sonic energy with frequency ω at the rate U (watts) The response of the receiver depends on the direction of the source with respect to the receiver This may be designated by an angle α It is important in practice to ascertain the directivity pattern of a receiver i e , curves showing how the response of the receiver varies with the relative direction of the source These curves resemble a cluster of flower petals Dimensional analysis alone provides no information concerning directivity patterns, since a direction is defined by dimensionless variables

The response of a receiver may be defined by the electromotive force ϕ of the alternating current that is generated by the vibrating crystals This is given by an equation of the following form

$$\phi = f(L, \alpha, U, \omega, \epsilon, x, \lambda, \rho, d) \qquad \text{(a)}$$

in which L is the distance from the source of sound to the instrument, α is the direction of the source with respect to the instrument, U is the power emitted by the source, ω is the frequency, x is a characteristic length of the instrument, ϵ is a characteristic coefficient of electric inductive capacity, λ is a characteristic elastic constant of the instrument, ρ is a characteristic mass density, and d is a characteristic piezoelectric coefficient. Various densities may enter the problem; for example, the densities of the oil, the water, and the crystals. However, if all instruments under consideration are made of the same materials, only a single representative density is needed. Likewise, if all instruments under consideration have the same shape, a characteristic length x specifies completely the geometry of an instrument. Velocities of sound in oil and in water need not be considered, since they are proportional to $\sqrt{\lambda/\rho}$.

The preceding variables furnish the following complete set of dimensionless products:

$$\alpha, \frac{x}{L}, \frac{\lambda d^2}{\epsilon}, \frac{U^2\rho}{\lambda^3 L^4}, \frac{\omega^2 x^2 \rho}{\lambda}, \frac{\phi d}{x}$$

Hence, by Buckingham's theorem,

$$\frac{\phi d}{x} = f\left(\alpha, \frac{x}{L}, \frac{\lambda d^2}{\epsilon}, \frac{U^2\rho}{\lambda^3 L^4}, \frac{\omega^2 x^2 \rho}{\lambda}\right) \tag{b}$$

This equation is valid for instruments of the same shape, the same materials, and the same orientations of crystal cuts. In this class of instruments, the quantities d, λ, ϵ, and ρ are constants. Consequently, Equation b is not essentially changed if it is written in the following simpler form:

$$\frac{\phi}{x} = f\left(\alpha, \frac{x}{L}, \frac{U}{L^2}, \omega x\right) \tag{c}$$

If spherical sound waves emanate from a source, the flux of sonic energy through a surface element at a distance L from the source is inversely proportional to the square of L. This follows directly from the law of conservation of energy. Consequently, if U is increased by a factor k^2 and L is increased by a factor k, the flux of sonic energy at the instrument is unchanged. Since the receiver responds only to the local flux of sonic energy, it follows that the term x/L actually does not enter Equation c. Therefore, the equation reduces to

$$\frac{\phi}{x} = f\left(\alpha, \frac{U}{L^2}, \omega x\right) \tag{d}$$

Some instruments have linear responses in limited ranges, i.e., for certain ranges of sound intensity, ϕ is proportional to U. Then Equation d takes the more special form,

$$\frac{\phi L^2}{Ux} = f(\alpha, \omega x) \tag{e}$$

It is well known that the output of a receiver may be expressed as a function of the direction α, the frequency ω, and the ratio U/L^2. The significant result of the preceding dimensional analysis is that it shows how the size variable x enters the problem. Hence, test data from a piezo electric receiver may be plotted in a form that is valid for similar instruments of different sizes. However, the instruments must not be too small with respect to the wave length, since there are scale effects that have been disregarded in the preceding analysis.

If the operation of a piezoelectric receiver is reversed, the instrument becomes a transmitter. A piezoelectric transmitter is ordinarily operated at the mechanical resonance frequency of the crystals. Since the amplitude of a resonating mechanical system is strongly affected by damping, a friction coefficient must be included in the list of variables in the dimensional analysis of a piezoelectric transmitter.

PROBLEMS

1. Prove by dimensional analysis that the heat W that is emitted per second from an electric heating coil is proportional to I^2R, in which I is the current and R is the resistance.

2. The electric energy U in a capacitor depends on the charge q and the capacitance C. How does U vary with q? With C?

3. The current I in a wire depends on the potential drop ϕ and the resistance R. What is the most general form of a dimensionally homogeneous equation for I? How does this compare with Ohm's law?

4. Perform a dimensional analysis of the equation

$$H = f(m, \mu, \epsilon, c, R, I)$$

in which the variables are respectively magnetic field intensity, mass, magnetic inductive capacity, electric inductive capacity, velocity of light, electrical resistance, and electric current.

5. An electric charge that is vibrating with a definite frequency emits radiant energy of a definite wave length. The energy U that is emitted per second depends on the wave length λ, the amplitude of vibration a, the charge q, the electric inductive capacity ϵ_0 of empty space, and the speed of light c. Derive the most general form of a dimensionally homogeneous equation for U. How does U vary with q? If U is proportional to a^2, how does U vary with λ?

6. The magnetic field intensity H due to a current in a long straight wire depends only on the current I and the distance y from the wire. Show by dimensional analysis that H is proportional to I and inversely proportional to y.

7. The capacitance C of a plate capacitor depends on the electric inductive capacity ϵ of the dielectric, the distance h between the plates, and the area A of a plate. Assuming

that C is proportional to A, show by dimensional analysis that C is proportional to ϵ and inversely proportional to h.

8. Express a resistance of 1 ohm in units of the Gaussian system: i.e., grams, centimeters, seconds, and CGS electrostatic units of charge (1 coulomb = 3×10^9 CGS electrostatic units of charge).

9. The shaft power W of a d-c motor depends on the magnetic flux density B of the field, the length L of the armature, the diameter D of the armature, the armature current I, and the angular speed n of the shaft. Using the fact that W is proportional to L, derive the general form of the relationship.

10. If the plates of a capacitor are suddenly connected by a conductor, the current may oscillate back and forth between the plates. The period t of an oscillation depends on the resistance R and the self-inductance $\mathfrak{L}$ of the conductor and on the capacitance C of the capacitor. Derive the general form of the equation for t. If the self-inductance is negligible, how does t vary with C? With R?

11. Show by dimensional analysis that, if the size of an electromagnet is changed in the same proportion as the current in the coil, the magnetic field intensity is unchanged at a point whose relative position with respect to the magnet is fixed.

CHAPTER 10

Differential Equations and Similarity

Dimensional analysis has been developed principally by British and American scientists In continental Europe, model laws have been derived almost exclusively from the differential equations that govern phenomena Differential equations occasionally provide a deeper insight into the laws of similarity than a mere knowledge of the variables that enter the problems —particularly, if laws of similarity for distorted models are sought On the other hand, the method of differential equations is restricted in its generality, since the differential equations that govern many phenomena (e g turbulent flow) are unknown

66 Model Law for Unsteady Motion of a Body in an Incompressible Viscous Fluid

In the first half of the nineteenth century, the question of the effect of the buoyancy and the resistance of air on the motion of the pendulum of a clock attracted much interest George G Stokes[28] derived the model law for pendulums in viscous fluids by means of the differential equations that are now known as the Navier Stokes equations

Stokes' work is of historical interest, since it seems to be the origin of the expression dynamic similarity" in scientific literature Also, it points out the significance of the ratio of viscosity to density which is now called "kinematic viscosity " Significantly, Stokes' work antedates, by more than thirty years, the work of Osborne Reynolds in calling attention to the fact that "Reynolds' number" is the criterion for determining whether or not fluid motions are dynamically similar

Stokes' analysis determines the general model law for flow of incompressible viscous fluids with fixed boundaries For simplicity, plane flow will be considered Then, if the pressure and the body force are eliminated from the Navier Stokes equations by the application of the differen-

[28] G G Stokes, On the Effect of the Internal Friction of Fluids on the Motion of Pendulums, *Trans Cambridge Phil Soc*, Vol IX Part 2, 1856 (Read Dec 9, 1850)

tial operator "curl," the following well-known equation is obtained:

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \tag{a}$$

in which u and v are the velocity components in the x and y directions, and ω is the local angular velocity (vorticity) of the fluid. The same differential equation applies for a model; i.e.

$$\frac{\partial \omega'}{\partial t'} + u'\frac{\partial \omega'}{\partial x'} + v'\frac{\partial \omega'}{\partial y'} = \nu'\left(\frac{\partial^2 \omega'}{\partial x'^2} + \frac{\partial^2 \omega'}{\partial y'^2}\right) \tag{b}$$

in which primes refer to the model.

It is desired to determine the necessary and sufficient conditions for the existence of scale factors K_L, K_t, etc., such that

$$\omega = \frac{\omega'}{K_\omega}, \qquad u = \frac{u'}{K_V}, \qquad v = \frac{v'}{K_V}, \qquad x = \frac{x'}{K_L}$$

$$y = \frac{y'}{K_L}, \qquad t = \frac{t'}{K_t}, \qquad \nu = \frac{\nu'}{K_\nu}$$

If these constants exist, Equation a may be written

$$\frac{K_t}{K_\omega}\frac{\partial \omega'}{\partial t'} + \frac{K_L}{K_V K_\omega}\left(u'\frac{\partial \omega'}{\partial x'} + v'\frac{\partial \omega'}{\partial y'}\right) = \nu'\frac{K_L^2}{K_\omega K_\nu}\left(\frac{\partial^2 \omega'}{\partial x'^2} + \frac{\partial^2 \omega'}{\partial y'^2}\right)$$

or

$$\frac{\partial \omega'}{\partial t'} + \frac{K_L}{K_V K_t}\left(u'\frac{\partial \omega'}{\partial x'} + v'\frac{\partial \omega'}{\partial y'}\right) = \nu'\frac{K_L^2}{K_t K_\nu}\left(\frac{\partial^2 \omega'}{\partial x'^2} + \frac{\partial^2 \omega'}{\partial y'^2}\right) \tag{c}$$

By Equation 12, the ratio $K_L/(K_V K_t)$ is invariably unity. Accordingly, Equations b and c show that kinematic and dynamic similarity of flows with similar boundary conditions are insured by the following condition:

$$\frac{K_L^2}{K_t K_\nu} = 1 \tag{d}$$

This is the relationship that Stokes derived. It is equally valid for steady and unsteady flows of liquids with fixed boundaries. Also, since the details of turbulent flow are probably governed by the Navier-Stokes equations, it may be expected to remain valid for turbulent flow.

Since $K_V = K_L/K_t$, Equation d may be alternatively expressed,

$$\frac{K_L K_V}{K_\nu} = 1 \tag{e}$$

This equation means that the scale factor for Reynolds' number is unity; i.e., if dynamic similarity exists, the Reynolds numbers of the model and the prototype must be equal This form of the law of similarity is commonly used in cases of steady flow

There is another common method for deriving laws of similarity from differential equations that is intrinsically the same as the foregoing method but differs from it formally The basic idea of the alternative method is to express the differential equations in dimensionless forms In the present case, this is accomplished by introducing a characteristic length L, a characteristic time period T, a characteristic velocity V, a characteristic angular velocity Ω, and a characteristic value N of the kinematic viscosity For example, in the case of periodic flow in a closed conduit, L might be the diameter at a certain section, T might be the period of a cycle, V might be the maximum velocity at a certain section, etc Dimensionless variables $\bar{x}$, $\bar{y}$, etc , may be defined as follows

$$x = \bar{x}L, \qquad y = \bar{y}L, \qquad t = \bar{t}T, \qquad u = \bar{u}V$$

$$v = \bar{v}V, \qquad \omega = \bar{\omega}\Omega, \qquad \nu = \bar{\nu}N$$

In terms of the new variables, Equation a is expressed as follows

$$\frac{\partial\bar{\omega}}{\partial\bar{t}} + \frac{VT}{L}\left(\bar{u}\frac{\partial\bar{\omega}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{\omega}}{\partial\bar{y}}\right) = \frac{NT}{L^2}\bar{\nu}\left(\frac{\partial^2\bar{\omega}}{\partial\bar{x}^2} + \frac{\partial^2\bar{\omega}}{\partial\bar{y}^2}\right) \tag{f}$$

If homologous points and homologous times are considered, the dimensionless variables $\bar{x}$, $\bar{y}$, $\bar{t}$, $\bar{u}$, $\bar{v}$, $\bar{\omega}$, $\bar{\nu}$ have the same values for a model and its prototype Hence, if the systems are similar, the coefficients VT/L and NT/L^2 must be the same for the two systems, i e ,

$$\frac{VT}{L} = \frac{V'T'}{L'}, \qquad \frac{NT}{L^2} = \frac{N'T'}{(L')^2}$$

in which the primes refer to the model Since $V'/V = K_V$, etc , these equations may be expressed

$$K_V K_t = K_L, \qquad K_\nu K_t = K_L^2$$

The first of these equations is automatically satisfied, by virtue of Equation 12 The second equation is identical to Equation d, which was derived by the other method

Of the two methods, the first has the advantage that it requires fewer notations Differential equations, of course, have no inherent advantage for the treatment of the foregoing problem The same results may be derived by the method of Chapters 2 and 3

67 Distorted Models of Rivers and Estuaries

Questions concerned with the releasing of waters that are impounded by dams, general problems of flood routing, questions of the effects of levees and cut-offs, and many other problems are studied by means of models of rivers and estuaries. A typical scale factor for the horizontal lengths in a river model is 1/1000, whereas the scale factor for vertical lengths is usually not less than 1/100. The most notable feature of a distorted model of a river is the steepening of the banks, bottom slopes, and other inclinations. Obviously, the scale factor for any slope is equal to the distortion factor. Since the bottom slopes are increased and the depths are relatively increased, the water tends to flow too rapidly in a distorted model. Consequently, it is necessary to roughen the walls of the model with corrugations or stucco, in order to retard the flow. Frequently, screen wire is used to simulate the roughness due to brush, weeds, and trees on the banks.

In all cases of flow with a free surface, equivalence of Froude numbers is a necessary condition for similarity. The Reynolds numbers are usually of secondary importance. Accordingly, the scaling of models of open channels is said to be governed by "Froude's law." However, with a distorted model, the question arises whether a horizontal length, a vertical length, or some weighted average of these two lengths should be employed for calculating Froude's number. Reynolds conjectured that a vertical length should be used, since the vertical length scale determines the relative wave speeds. The equivalence of Froude numbers of a river model and its prototype is accordingly expressed by the equation, $(V')^2/L'g = V^2/Lg$, in which L' and L are vertical lengths of the respective systems. Hence, the scale factor K_z for vertical lengths is related to the scale factor K_V for horizontal velocities by the equation,

$$K_V = \sqrt{K_z} \tag{a}$$

Also, by Equation 12, $K_x/K_t = K_V$, in which K_x is the scale factor for horizontal lengths and K_t is the time scale factor. It follows that

$$K_t = \frac{K_x}{\sqrt{K_z}} \tag{b}$$

For example, let $K_x = 1/1000$ and $K_z = 1/50$. Then, by Equations a and b, $K_V = 0.142$ and $K_t = 0.00707$; i.e., the horizontal velocities in the model are about 14 percent of the corresponding velocities in the prototype, and the time periods in the model are about 0.71 percent of the corresponding periods in the prototype. For a model of an estuary with these proportions, the period from high tide to high tide is about 5 min. The tide-generating machine must be adjusted to give the proper tidal period for the model. The velocities then automatically assume the correct values, if the roughnesses of the banks and the bottom are correct. The adjustment of the roughnesses is a trial procedure that usually consumes a large amount of time.

A rigorous analysis of distorted models of open channels has been presented by A T Doodson[29] Doodson based his argument on the continuity equation and the momentum equation If the velocity V of the current is assumed to be constant at any instant on any cross section of the stream, the continuity equation is

$$b\frac{\partial z}{\partial t} + \frac{\partial (VA)}{\partial x} = 0 \tag{c}$$

in which x is a horizontal coordinate in the downstream direction, z is the elevation of the free surface at section x (with reference to a horizontal datum plane), b is the width of the free surface at section x, and A is the cross-sectional area of the stream at section x This equation merely expresses the fact that the net rate of flow into the region between cross sectional planes with coordinates x and $x + dx$ is the rate at which the volume of water in this region is increasing

The momentum equation expresses the fact that the net force on the slab of fluid between cross-sectional planes with coordinates x and $x + dx$ is equal to the net rate at which momentum is convected out of the region between these planes plus the time rate of change of momentum The momentum equation may be expressed

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} = -g\frac{\partial z}{\partial x} \pm \frac{kV^2}{R} \tag{d}$$

in which R is the hydraulic radius at section x and $k\rho V^2$ is the average shearing stress on the banks and the bottom at section x Although k depends on Reynolds' number, there is only a small error in assuming that k is constant

If Equation c and d could be solved, subject to given time dependent boundary conditions for the sections $x = 0$ and $x = x_0$, the flow at any time and at any section in the interval $(0, x_0)$ would be determined However, this is a very difficult problem

On the other hand, Doodson has shown that it is easy to deduce the law of similarity for models of open channels by means of Equations c and d To this end, the following substitutions are introduced

$$x = \frac{x'}{K_x}, \qquad z = \frac{z'}{K_z}, \qquad t = \frac{t'}{K_t}, \qquad b = \frac{b'}{K_x}$$

$$A = \frac{A'}{K_x K_z}, \qquad V = \frac{V'}{K_V}, \qquad R = \frac{R}{K_R}, \qquad k = \frac{k'}{K_k}$$

[29] A T Doodson Tide Models *Dock and Harbour Authority*, Vol XXIX, no 339, p 223 Jan 1949 Doodson's analysis is reproduced here through the courtesy of the Dock and Harbour Authority

in which primes refer to the model. Substitution of these expressions in Equations c and d yields

$$\frac{K_t}{K_x K_z} b' \frac{\partial z'}{\partial t'} + \frac{1}{K_V K_z} \frac{\partial (V'A')}{\partial x'} = 0 \tag{c'}$$

$$\frac{K_t K_V}{K_x} \frac{\partial V'}{\partial t'} + V' \frac{\partial V'}{\partial x'} = -g \frac{K_V^2}{K_z} \frac{\partial z'}{\partial x'} \pm \frac{K_R}{K_k K_x} k' \frac{V'^2}{R'} \tag{d'}$$

If the scale factors exist, Equations c′ and d′ must reduce, respectively, to Equations c and d, except for the primes on the terms. Equation c′ automatically satisfies this condition, since the relationship $K_t = K_x/K_V$ follows from Equation 12. Equation d′ accordingly may be expressed,

$$\frac{\partial V'}{\partial t'} + V' \frac{\partial V'}{\partial x'} = -g \frac{K_V^2}{K_z} \frac{\partial z'}{\partial x'} \pm \frac{K_R}{K_k K_x} k' \frac{V'^2}{R'}$$

This reduces to the form of Equation d if, and only if,

$$\frac{K_V^2}{K_z} = 1 \tag{e}$$

and

$$\frac{K_R}{K_k K_x} = 1 \tag{f}$$

Equation e is the result that was previously derived by Froude's law. Thus, the basing of Froude's number on a vertical length is justified. Equation f yields

$$K_k = \frac{K_R}{K_x} \tag{g}$$

For any given shape of cross section, K_R is expressible in terms of K_x and K_z by means of geometrical relationships. For a wide shallow channel, K_R is practically equal to K_z. Accordingly, Equation g shows that K_k is approximately equal to the distortion factor. Since k cannot be increased indefinitely, there is consequently a limit to the amount of distortion that a model may have, if it is to operate satisfactorily. Experience with models of rivers and estuaries indicates that the friction coefficient k automatically increases to some extent as the distortion factor increases. This is possibly due to the fact that turbulence and eddying are increased by distortion.

68. Model Law for Electromagnetic Phenomena

A typical general boundary value problem of electromagnetic theory is the following: In a system of conductors and dielectrics, the initial values of the field vectors E and H are given, and, on a certain boundary, the tangential components of E are prescribed functions of time. It is required to derive the solution of Maxwell's equations that satisfies the initial conditions and the boundary conditions.

If the materials are isotropic and nonmagnetic, Maxwell's equations may be expressed as follows

$$\text{curl } E + \mu \frac{\partial H}{\partial t} = 0 \tag{a}$$

$$\text{curl H} - \epsilon \frac{\partial E}{\partial t} = \kappa E \tag{b}$$

Consider two geometrically similar systems that have similar electromagnetic properties, i.e. similar distributions of ϵ, μ, and κ. Furthermore, let the initial conditions and the boundary conditions for the two systems be similar.

If one of the systems (called the model) is designated by primes, its behavior is governed by the differential equations,

$$\text{curl}' E' + \mu' \frac{\partial H}{\partial t'} = 0 \tag{a'}$$

$$\text{curl}' H' - \epsilon' \frac{\partial E'}{\partial t} = \kappa' E' \tag{b'}$$

in which the notation curl′ indicates that derivatives are taken with respect to the accented coordinates. If the scale factors, $K_E = E'/E$, etc., are introduced, Equations a and b may be alternatively written

$$\frac{K_L}{K_E} \text{curl}' E' + \frac{K_t}{K_\mu K_H} \mu' \frac{\partial H'}{\partial t'} = 0 \tag{c}$$

$$\frac{K_L}{K_H} \text{curl}' H' - \frac{K_t \epsilon'}{K_\epsilon K_E} \frac{\partial E'}{\partial t'} = \frac{1}{K_\kappa K_E} \kappa E \tag{d}$$

In order that Equations c and d shall reduce respectively to Equations a′ and b′, it is necessary and sufficient that

$$\frac{K_E K_t}{K_\mu K_H K_L} = 1, \qquad \frac{K_H K_t}{K_\epsilon K_E K_L} = 1, \qquad \frac{K_H}{K_\kappa K_E K_L} = 1 \tag{e}$$

Equations e yield

$$K_\kappa = \frac{1}{K_L}\sqrt{\frac{K_\epsilon}{K_\mu}}, \qquad K_t = K_L\sqrt{K_\epsilon K_\mu}, \qquad \frac{K_H}{K_E} = \sqrt{\frac{K_\epsilon}{K_\mu}} \tag{f}$$

Equation f shows that, when K_L is prescribed, the factors K_ϵ, K_μ, and K_κ may not be assigned independently. Conversely, if the model and the prototype are made of the same materials (i.e., $K_\epsilon = K_\mu = K_\kappa = 1$), then $K_L = 1$. Accordingly, if it is desired that the model and the prototype shall have different sizes, the two systems may not be made of the same materials. The equation $K_t = K_L\sqrt{K_\epsilon K_\mu}$ signifies that the velocity scale factor is identical to the scale factor for the speed of electromagnetic waves, since the speed of an electromagnetic wave is $1/\sqrt{\epsilon\mu}$.

69. Interpretation of Dimensionless Products in Fluid Mechanics

In fluid mechanics, the total force on a particle is decomposed into gravity force F_g, viscous friction force F_f, pressure force F_p, and inertia force F_i. Each of these forces is represented by a term in the Navier-Stokes equations. Writing the first of the Navier-Stokes equations for a steady-flow process and multiplying by a volume element dQ, we get

$$\rho g \cos\alpha\, dQ - \frac{\partial p}{\partial x}\, dQ + \mu\, \nabla^2 u\, dQ - \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) dQ = 0$$

The four terms in this equation are, respectively, the x-components of the gravity force, the pressure force, the viscous friction force, and the inertia force on the particle dQ; i.e.,

$$F_g = \rho g \cos\alpha\, dQ, \qquad F_p = -\frac{\partial p}{\partial x}\, dQ$$

$$F_f = \mu\nabla^2 u\, dQ$$

$$F_i = -\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) dQ$$

In a class of geometrically similar systems with kinematically similar flows, a velocity field is determined by a single velocity V that specifies the speed of flow, and a single length L that specifies the size of the system. Set

$$x = L\bar{x}, \qquad y = L\bar{y}, \qquad z = L\bar{z}$$

$$u = V\bar{u}, \qquad v = V\bar{v}, \qquad w = V\bar{w}$$

Then $(\bar{x}, \bar{y}, \bar{z})$ and $(\bar{u}, \bar{v}, \bar{w})$ are dimensionless.

Let us also restrict attention to systems with similar pressure distributions.* Then, $p = P\bar{p}$, in which $\bar{p}$ is dimensionless. Accordingly, the

* Since an arbitrary constant pressure may be impressed on a system without altering the flow, the condition of similar pressure distributions is not necessary for kinematic similarity.

equations for F_g, F_p, F_f, and F_i may be written,

$$F_g = \rho g \cos \alpha \, dQ$$

$$F_p = -\frac{P}{L}\frac{\partial \bar{p}}{\partial \bar{x}} dQ$$

$$F_f = \frac{\mu V}{L^2}\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2}\right) dQ$$

$$F_i = -\frac{\rho V^2}{L}\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{u}}{\partial \bar{z}}\right) dQ$$

Accordingly, the ratios of these forces are

$$\frac{F_i}{F_f} = -\mathrm{R}\frac{\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{u}}{\partial \bar{z}}}{\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2}}$$

$$\frac{F_p}{F_i} = \mathrm{P}\frac{\frac{\partial \bar{p}}{\partial \bar{x}}}{\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{u}}{\partial \bar{z}}}$$

$$\frac{F_i}{F_g} = -\mathrm{F}\frac{\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{u}}{\partial \bar{z}}}{\cos \alpha}$$

in which $\mathrm{R} = \frac{VL\rho}{\mu}$ = Reynolds' number

$\mathrm{P} = \frac{P}{\rho V^2}$ = pressure coefficient

$\mathrm{F} = \frac{V^2}{Lg}$ = Froude's number

The factors containing derivatives are dimensionless, and they consequently have the same values for all systems Accordingly, the following principle is established

In geometrically similar systems with kinematically similar steady flows and similar pressure distributions, the ratios of inertia force to friction force are identical if the Reynolds numbers are equal, the ratios of inertia force to pressure force are identical if the pressure coefficients are equal, *and the ratios of inertia force to gravity force are identical if the Froude numbers are equal.*

70. Differential Theorem on Dimensional Homogeneity

In Article 19, it was demonstrated that an equation $y = f(x_1, x_2, \cdots, x_n)$ is dimensionally homogeneous if, and only if, the relationship,

$$Kf(x_1, x_2, \cdots, x_n) = f(K_1x_1, K_2x_2, \cdots, K_nx_n) \qquad \text{(a)}$$

is an identity in the variables $x_1, x_2, \cdots, x_n, A, B, C$, in which

$$K = A^aB^bC^c$$

and

$$K_i = A^{a_i}B^{b_i}C^{c_i}, \qquad i = 1, 2, \cdots, n$$

where A, B, C are any positive numbers, and the dimensional matrix is

	y	x_1	$x_2 \cdots x_n$
M	a	a_1	$a_2 \cdots a_n$
L	b	b_1	$b_2 \cdots b_n$
T	c	c_1	$c_2 \cdots c_n$

This theorem establishes a relation between the mathematical use of the word "homogeneity" and the use of this word in dimensional analysis. For, if the variables $(x_1, x_2, \cdots, x_n)$ all have the same dimension and if the dimensional exponents of y are proportional to the dimensional exponents of the x's, Equation a reduces to the simpler form,

$$C^Nf(x_1, x_2, \cdots, x_n) \equiv f(Cx_1, Cx_2, \cdots, Cx_n)$$

in which the symbol $\equiv$ signifies that the equality is an identity in the variables $(x_1, x_2, \cdots, x_n, C)$. Functions with this property were studied by Euler. They are known as "homogeneous functions" (more specifically, functions that are homogeneous of degree N). Equation a consequently shows that dimensionally homogeneous functions are a generalization of the homogeneous functions of Euler.

Euler derived a differential theorem concerning homogeneous functions, which is readily generalized to include the class of dimensionally homogeneous functions. To derive this theorem, we write Equation a in the form,

$$\bar{y} = f(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \qquad \text{(b)}$$

in which $\bar{y} = Ky, \qquad \bar{x}_i = K_ix_i, \qquad i = 1, 2, \cdots, n$

Differentiation of Equation b with respect to A yields

$$\frac{\partial \bar{y}}{\partial A} = \frac{\partial \bar{y}}{\partial \bar{x}_1}\frac{\partial \bar{x}_1}{\partial A} + \frac{\partial \bar{y}}{\partial \bar{x}_2}\frac{\partial \bar{x}_2}{\partial A} + \cdots + \frac{\partial \bar{y}}{\partial \bar{x}_n}\frac{\partial \bar{x}_n}{\partial A} \qquad \text{(c)}$$

Let us introduce the above values of $\bar{y}, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ in Equation c, perform the indicated differentiations, and then set $A = B = C = 1$. This yields

$$ay = a_1x_1\frac{\partial y}{\partial x_1} + a_2x_2\frac{\partial y}{\partial x_2} + \cdots + a_nx_n\frac{\partial y}{\partial x_n}$$

Similarly, two other equations are derived by assigning to B and C the role of A in the above derivation. Thus, the following theorem is proved:

If a function $y = f(x_1, x_2, \ldots, x_n)$ is differentiable and dimensionally homogeneous, it is a solution of the differential equations,

$$\begin{aligned} ay &= a_1x_1\frac{\partial y}{\partial x_1} + a_2x_2\frac{\partial y}{\partial x_2} + \cdots + a_nx_n\frac{\partial y}{\partial x_n} \\ by &= b_1x_1\frac{\partial y}{\partial x_1} + b_2x_2\frac{\partial y}{\partial x_2} + \cdots + b_nx_n\frac{\partial y}{\partial x_n} \\ cy &= c_1x_1\frac{\partial y}{\partial x_1} + c_2x_2\frac{\partial y}{\partial x_2} + \cdots + c_nx_n\frac{\partial y}{\partial x_n} \end{aligned} \tag{35}$$

The converse of this theorem may also be proved, i.e.:

If a function satisfies Equation 35, *it is dimensionally homogeneous.*

However, the proof of this theorem will be omitted, since it cannot be accomplished without recourse to the general theory of linear first order partial differential equations.

In the class of functions that possess first derivatives, Equations 8 and 35 are completely equivalent. Observe that these equations are identically satisfied, if the variables are dimensionless.

Problems

1. If a flat elastic plate vibrates freely, the deflection w at the time t and at the point (x, y) of the middle plane is determined by the differential equation,

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} + C\frac{\partial^2 w}{\partial t^2} = 0$$

in which C is a constant that depends on the elastic properties of the material, the density of the material, and the thickness of the plate. Prove that a model of the plate that preserves kinematic similarity can have no distortion of planform. Prove that, when the planforms of two plates are geometrically similar, K_t does not depend on K_w. Hence express K_t as a function of K_x and K_C. If $K_x = 1/3$ and $K_t = 1/10$, what is the value of K_C?

2. Prove the theorem on dimensional homogeneity of a sum of terms (Article 20) by means of the theorem in Article 70.

3. The equation $y = f(x)$ is dimensionally homogeneous. By means of the theorem in Article 70, determine the most general form of this equation.

4. Derive Theorem 2 of Article 21 from the theorem of Article 70.

5. It has been shown (Article 6) that the drag force on a sphere in a stream of fluid is given by an equation of the form

$$F = \rho V^2 D^2 f\left(\frac{VD}{\nu}\right)$$

Show that this equation satisfies Equation 35.

6. Free vibration of a mass m with linear damping is governed by the differential equation $m\ddot{x} + c\dot{x} + kx = 0$, in which the dots denote time derivatives. Derive the model law of the phenomenon from this differential equation. Show that the same result can be obtained by the method of Chapter 5.

Answers to Problems

Chapter 1

2. 49.09 slug, 716,400 g, 1579 lb.
3. 95,500 asm, 4.08×10^{23} slug, 6.05×10^{23} kg sec²/m, sp gr = 5.50.
4. 114.
5. 3.24×10^{-16}.
6. $W = \dfrac{1.185\, sh^3}{(s-1)^3(\cos\alpha - \sin\alpha)^3}$, $W = \dfrac{0.0185\, wsh^3}{(s-1)^3(\cos\alpha - \sin\alpha)^3}$, 18,200 lb.
7. $C = (1.042 + 0.000129V^2)A$.
8. $M = 0.0317A\sqrt{\dfrac{(p_1 - p_2)p_2}{\theta_1}}$, $M = \dfrac{0.0158Ap_1}{\sqrt{\theta_1}}$.
9. $[L] = [T]$, [velocity] = [1], $[F] = [MT^{-1}]$, unit of force = 3×10^{10} dynes = 67,400 lb.

Chapter 2

2. $\Delta p = \dfrac{K\mu LV}{D^2}$, in which K is a dimensionless constant.
5. $y_{cr} = K\sqrt[5]{\dfrac{Q^2}{g}}$, in which K is a function of α.
6. $y_{cr} = K\sqrt[3]{\dfrac{q^2}{g}}$, in which K is a dimensionless constant.
7. $M = bh^2\sigma_y f\left(\dfrac{b}{h}\right)$, $M = Kbh^2\sigma_y$, in which K is a dimensionless constant.
8. $\dfrac{p_0}{\rho_0 V^2}$.
9. Reynolds' number, Mach's number, and angle of attack.
10. $\Delta p = \rho V^2 f(\mathrm{R})$.
11. h is proportional to σ and inversely proportional to w.
12. $h = Df\left(\dfrac{L}{D}, \dfrac{\tau}{wD}\right)$.
13. 1140 ft/sec.
15. Speed of sound is proportional to the square root of the modulus of elasticity and inversely proportional to the square root of the density.
16. n is inversely proportional to L, proportional to the square root of E, and inversely proportional to the square root of ρ.
18. $\eta = f\left(\dfrac{d}{D}, \dfrac{\mu ND}{F}\right)$.
19. $W = wD^3 f\left(\dfrac{wD^2}{\quad}\right)$.

21 $n = \frac{V}{D} f\left(\frac{VD\rho}{\mu}\right)$.

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23 V is proportional to the square root of T and inversely proportional to the square root of m

24 $Q = \sqrt{gh^5} f\left(\frac{h}{L}, \frac{e}{L}, \frac{\mu^2}{g\rho^2 L^3}\right)$

25 L is inversely proportional to V, directly proportional to μ, and inversely proportional to ρ

26 d is proportional to L^3 and inversely proportional to EI

27 $E = \frac{\rho Q^2}{D^4} f\left(\frac{\rho Q}{\mu D}\right)$

28 $k = Df\left(\frac{VD\rho_a}{\mu_a}, \frac{V^2}{gD}, \frac{\rho V^2 D}{\sigma}, \frac{\rho_a}{\rho}\right)$ in which k is the height to which the drops rise, D is the diameter of the jet V is the velocity of the jet σ is the surface tension of the liquid, ρ_a and ρ are the mass densities of air and of the liquid, and μ_a is the viscosity of air

Chapter 3

2 $k_4 = \frac{1}{3}(a + 2b + 3c)$, $k_5 = -\frac{1}{6}(a + 2b + 6c)$, $k_6 = \frac{1}{3}(b - a)$, $r = c$, $s = \frac{1}{6}(a + 2b + 6c)$, $t = \frac{1}{3}(a - b)$

4 $QP^{-2}M^2N^4$ $AP^{-1}M$, $VP^{-1}MN$

5 $r = 3$, $A^4D^{14}EF^4$ B^2EF, $C^2D^6E^{-1}F^{-1}$

6 $r = 3$, $A^{13}F^{-1}G^{23}H^{7}$ $B^{16}F^{-3}GH^{-11}$ C, $D^2F^3G^{-3}H^{-1}$, $E^{12}F^{10}G^{11}H^2$

7 $r = 2$, $A^{16}C^7D^9$, $B^5C^6D^2$

8 $r = 4$, $A^{36}D^{15}E^{-149}F^{-56}G^{77}$, $B^{38}D^{67}E^{-315}F^{-96}G^{17}$, $C^7D^{-8}E^{24}F^9G^{-3}$

9 $r = 2$, AE^3, BE^2F^{-2} CEF^{-1}, DE^3F^{-1}

10 $r = 2$ $A^4C^{-2}D^{-7}$, B^4C^2D

11 $V = \sqrt[3]{\frac{\mu_a g}{\rho_a}} f\left(\frac{\rho_a}{\rho_w}, \frac{\mu_a}{\mu_w}\right)$

12 $V = \frac{\sigma}{\mu_w} f\left(\frac{h\rho_w\sigma}{\mu_w^2}, \frac{\mu_a}{\mu_w}, \frac{\rho_a}{\rho_w}\right)$

13 $h = Hf\left(\frac{Q}{\sqrt{gH^5}}, n\sqrt{\frac{H}{g}}, \frac{\nu}{\sqrt{gH^3}}, \alpha\right)$ where $\nu = \frac{\mu}{\rho}$

14 $M = \rho g L^4 f\left(\frac{V^2}{Lg}, \frac{m}{\rho L^3}, \frac{R}{L}, \alpha, \beta\right)$

16 $\frac{pW'}{\mu^2}$, $VD\sqrt{\frac{p}{W'}}$

20 $\frac{\sigma\rho D}{\mu^2} = f\left(\frac{w\mu^4}{\sigma^3\rho^2}\right)$, D = diameter σ = surface tension, w = specific weight of liquid, ρ = mass density of gas, μ = viscosity of gas

21. Specific weight w of raindrop, diameter D of raindrop, surface tension σ of raindrop, mass density ρ of air, viscosity μ of air.

$$V = \sqrt{\frac{\sigma}{D\rho}}\, f\left(\frac{wD^2}{\sigma}, \frac{D\sigma\rho}{\mu^2}\right).$$

22. Velocity V of wind, diameter D of rod, length L of rod, modulus of elasticity E of rod, mass density ρ_a of air, mass density ρ_r of rod, viscosity μ_a of air.

$$\delta = Df\left(\frac{L}{D}, \frac{VD\rho_a}{\mu_a}, \frac{\rho_a}{\rho_r}, \frac{E}{\rho_a V^2}\right).$$

23. A characteristic length L of the airplane, diameter D of a raindrop, number N of raindrops per unit volume, mass density ρ_r of water, surface tension σ of water, acceleration of gravity g, mass density ρ_a of air, viscosity μ_a of air, speed V of airplane.

$$n = VNL^2 f\left(\frac{L}{D}, \frac{\rho_a}{\rho_r}, \frac{VL\rho_a}{\mu_a}, NL^3, \frac{V^2}{gD}, \frac{\sigma}{\rho_r V^2 D}\right).$$

24. Velocity V of wind, diameter D of a sand grain, specific weight w of sand, mass density ρ of air, viscosity μ of air.

$$W = \rho V^2 f\left(\frac{VD\rho}{\mu}, \frac{wD}{\rho V^2}\right).$$

25. Diameter D of the propeller, rotational speed n of the propeller, the distance L, mass density ρ of air, pressure p of air.

$$U = pf\left(\frac{L}{D}, nD\sqrt{\frac{\rho}{p}}\right).$$

26. The mass m of the drop, the velocity V of the drop, the surface tension σ of the liquid, the mass density ρ of the liquid, the viscosity μ of the liquid, the acceleration of gravity g.

$$h = \sqrt[3]{\frac{\mu^2}{g\rho^2}}\, f\left(\frac{m\rho g}{\mu^2}, \frac{\rho V^3}{\mu g}, \frac{\rho\sigma^3}{g\mu^4}\right).$$

Chapter 4

2. The products are not independent, since the third row in the matrix of exponents is a linear combination of the other two rows.
3. No.
4. The equations are inconsistent.
5. $x^{1/3}y^{2/3}z^{-1}$. A product with the dimension $[MLT]$ does not exist.
6. 0.000395.

Chapter 5

1. 1/3. Flows are completely similar.
2. 28.5 lb, 12.2 ft/sec.
3. 94.9 mi/hr.
4. 16,000 lb.
5. 15.8.
6. 0.00476 slug/ft³.
7. $d = f\left(\frac{I}{mL^2}, \frac{m}{L^3}, \frac{v^2}{L^2}\right).$

8 Distortion factor = 10 $K_u = K_v = 0.12$ $K_w = 1.20$ $K_{a_x} = K_{a_y} = 14.4$ $K_{a_z} = 144$ $K_{F_x} = K_{F_y} = 1.44 \times 10^{-7}$, $K_{F_z} = 1.44 \times 10^{-6}$ $K_T = 0.127$

9 11.98 ft/sec

10 2.5×10^{-6}

11 12 ft³/sec 0.5625 lb/in²

12 2 in./min 312 500 lb

13 $K_V = 0.447$ $K_t = 0.447$ $K_a = 1.00$ $K_F = 0.008$ $K_{a_1} = 2.236$ 4.472 sec

14 $K_L = 0.600$ $K_V = 1.667$ $K_t = 0.360$ 0.018 in 66.7 ft/sec 3 in $K_{vol} = 0.216$.

15 $K_{spring} = 4.00$ $K_{damping} = 0.050$

16 $K_1^{\,3}K_2^{\,4} = K_3^{\,3}K_4^{\,4}$

17 Area 1/64 volume 1/512 weight 1/512 force 1/512 moment of force 1/4096 moment of inertia 1/32 768 velocity $\sqrt{2}/4$ acceleration 1 angular velocity $2\sqrt{2}$ angular speed of propeller $2\sqrt{2}$ horsepower $\sqrt{2}/2048$ time $\sqrt{2}/4$ 195 lb 5.54 hp

Chapter 6

2 160 000 lb 1.20 in

3 16 875 lb in

4 As the square of the diameter

5 $x = \frac{2}{\pi}\left[y \arcsin\frac{1}{y} + \frac{1}{3}\left(5 - \frac{2}{y^2}\right)\sqrt{1 - \frac{1}{y^2}}\right]$ for $y > 1$

and $x = y$ for $y < 1$ where $x = \frac{Mc}{EI\epsilon_y}$ and $y = \frac{c}{r\epsilon_y}$

The required curve is the graph of this function in the interval $0 < x < \frac{16}{3\pi}$

7 83.6%

8 66.3%

9 60%

11 592 cycles/sec

12 $K_A = K_L$, $K_P = K_E K_L^2$ $K_n^2 K_L^2 K_\rho = K_E$
$K_A = 0.20$ $K_P = 0.020$ $K_n = 6.125$

14 They are directly proportional to Young's modulus The membrane energy is proportional to the thickness the bending energy is proportional to the cube of the thickness and the shear energy is proportional to the fifth power of the thickness

15 $r = \sigma_y f\left(\frac{h}{b}, \frac{d}{t}, \frac{b}{t}, \frac{\sigma_y}{E}\right)$

Chapter 7

1 $v^* = 2.50$ ft/sec $\tau_0 = 0.0149$ lb/ft² 25.6 ft/sec 54.4 ft/sec

2 $v^* = 1.19$ ft/sec $\tau_0 = 0.00337$ lb/ft² 33.1 ft/sec 46.8 ft/sec $\epsilon = 0.00155$ ft

5 $V = 11.7$ ft/sec R = 1 060 000 $v^* = 0.622$ ft/sec $u_{max} = 13.9$ ft/sec

6 13.42 14.95 16.47 17.99 18.88 19.51 ft/sec

7 2133 gal/min 1.46

8 Abscissa = $nD\sqrt{\frac{\rho}{p}}$, ordinate = $\frac{Q_{max}}{D^3}\sqrt{\frac{\rho}{p}}$

9 $n = \frac{1}{D}\sqrt{\frac{P}{\rho}}\, f\left(\frac{l}{D}\sqrt{\frac{P}{\rho}}, \frac{1}{n_0 D}\sqrt{\frac{P}{\rho}}\right)$

10. $D = \frac{1}{n}\sqrt{\frac{P}{\rho}} f_1\left(n\sqrt[4]{\frac{E^2\rho^3}{P^5}}\right),$

$$Q = \frac{P^{3/2}}{n^2\rho^{3/2}} f_2\left(n\sqrt[4]{\frac{E^2\rho^3}{P^5}}\right), \quad \eta = f_3\left(n\sqrt[4]{\frac{E^2\rho^3}{P^5}}\right).$$

11. $M' = 4M, \quad p = p', \quad E' = 4E.$

14. $C_f = 1.328R_x^{-1/2}.$

15. 0.14 ft, 42,080 lb, 2295 hp.

16. 50,300 ft³/sec, 2.50 days.

17. The difference p between the static pressure at the shaft level and the vapor pressure, the diameter D of the propeller, the speed V of the ship, the mass density ρ of water, and the acceleration of gravity g.

$$n_c = \frac{1}{D}\sqrt{\frac{p}{\rho}} f\left(\frac{1}{V}\sqrt{\frac{p}{\rho}}, \frac{V^2}{gD}\right).$$

$$K_V = \sqrt{K_D}, \quad K_p = K_D, \quad K_{n_c} = \frac{1}{\sqrt{K_D}}.$$

Chapter 8

2. $K = f(R, Q).$

3. $\frac{hL}{k} = f\left(\beta\theta, \frac{k\theta}{\rho L g^3 t^2}, \frac{gt^2}{L}, \frac{C\theta t^2}{L^2}\right).$

4. $\eta = f\left(\frac{\theta_2}{\theta_1}\right), \quad \eta = 1 - \frac{\theta_2}{\theta_1}.$

6. $\frac{E\nu^4 w^3 \beta^4}{k^4} = f\left(\frac{C\rho\nu}{k}\right).$

7. $N = f(R, Q).$

8. The diameter D, the length L, the temperature difference $\Delta\theta$, the heat capacity per unit volume $C\rho$, the thermal conductivity k of the fluid, the coefficient of thermal expansion β of the fluid, the kinematic viscosity ν of the fluid, and the acceleration of gravity g.

$$\frac{hD}{k} = f\left(G, Q, \frac{L}{D}\right).$$

9. The difference between the temperature of the fluid and the initial temperature of the ball $\theta_f - \theta_0$, the thermal conductivity k of the ball, the heat-transfer coefficient h of the surface, the heat capacity of the ball per unit volume $C\rho$, the diameter D of the ball, the elapsed time t, and the coefficient of thermal expansion β of the ball.

$$\epsilon = f\left[\frac{kt}{C\rho D^2}, \frac{hD}{k}, \beta(\theta_f - \theta_0)\right].$$

10. 41.9 min

Chapter 9

2 U is proportional to q^2 and inversely proportional to C

3 $I = \dfrac{K\phi}{R}$.

4 $\dfrac{Hmc^2}{I^2R} = f\left(\dfrac{\mu c}{R}, \epsilon cR\right)$.

5 $U = \dfrac{cq^2}{\epsilon_0 a^2} f\left(\dfrac{\lambda}{a}\right)$.

8 1 ohm $= 1.111 \times 10^{-12}\left[\dfrac{\text{g cm}^2}{\text{sec esu}^2}\right]$ "esu" denotes the electrostatic unit of charge

9 $W = KBInLD$ in which K is a constant

10 $t = CRf\left(\dfrac{CR^2}{\mathfrak{L}}\right)$

Chapter 10

1 $K_l = K_x^3 \sqrt{K_C}$, $K_C = 0.81$

3 $y = Kx^n$, in which n and K are dimensionless

6 $K_m = K_e K_l$, $\Lambda_m = \Lambda_k \Lambda_l^2$

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